On Consumer Theory with Indivisible Goods*

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Abstract

Baldwin and Klemperer (2019) classified valuations over indivisible goods into "demand types" by taking a particular geometric approach to analyze preferences. They showed that demand types have important economic properties related to the existence of competitive equilibrium with indivisible goods—leading to many novel domains for equilibrium existence. This paper shows how demand types can in fact be defined more conventionally in terms of simple conditions on the comparative statics of demand.

*This paper extends and supersedes much of Sections 4.2 and 5 of Baldwin and Klemperer's 2014 working paper "Tropical geometry to analyse demand."

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1 Introduction

Auctions often feature the simultaneous sale of multiple, indivisible goods. Yet, the perfect divisibility of goods is critical to analyzing preferences using standard, calculus-based methods.

Baldwin and Klemperer (2019, hereafter BK19) therefore developed new methods for analyzing quasilinear preferences over indivisible goods and money by introducing a geometric representation of such preferences. Calling the set of price vectors at which demand is nonunique the "locus of indifference prices" (LIP), they observe that the normal vector to a component of a LIP gives the change in demand as prices cross (only) that component. They then classified valuations into "demand types": each demand type is defined by a set of vectors that give the possible normal vectors to components of LIPs of that demand type.

BK19 showed that there are demand types corresponding to standard domains of preferences such as the class of all substitutes valuations, and the class of all complements valuations. They also showed that other demand types have important economic properties related to the existence of competitive equilibrium with indivisible goods: demand types that are *unimodular* correspond to domains for equilibrium existence (see also Danilov et al. (2001)). There are unimodular demand types corresponding to many previously known domains for equilibrium existence;[1] most unimodular demand types introduced novel domains.

This paper shows how all demand types can in fact be defined more conventionally in terms of simple conditions on the comparative statics of demand—thereby connecting demand types to classical consumer theory. More precisely, we show that interpreting the defining vectors of a demand type as the building blocks for changes in demand yields a simple, equivalent definition of demand types that does not rely on understanding the geometry of LIPs.

Consider first the case in which at most one unit of each good can be demanded. In this case, we show that a valuation is of a demand type if and only if demand always changes by a defining vector of the demand type when the price of one good is increased (Theorem 1). For example, if an agent wants to buy up to one unit of

[1]For example, BK19 showed that there are unimodular demand types corresponding to the class of "strong substitutes" valuations (Kelso and Crawford, 1982; Gul and Stacchetti, 1999, 2000; Milgrom and Strulovici, 2009), and to the class of "(generalized) gross substitutes and complements" valuations (Sun and Yang, 2006, 2009; Shioura and Yang, 2015).

each of two goods and views the goods as independent, then changing the price of one good can only change demand by $\pm(1,0)$ or $\pm(0,1)$. Thus, such an agent's valuation is of demand type $\pm\{(1,0),(0,1)\}$. If instead the agent saw the goods as substitutes, as increasing the price of one good could then make the agent substitute toward the other, changing the price of one good could also change demand by $(1,-1)$ or $(-1,1)$; in this case, the agent's valuation would be of demand type $\pm\{(1,0),(0,1),(1,-1)\}$.[2] Beyond these simple two-good cases (but maintaining the assumption that at most one unit of each good can be demanded), it follows from Theorem 1, for example, that an agent's valuation is of demand type $\mathcal{D} = \pm\{(1,1,0),(0,0,1),(1,1,-1)\}$ if and only if increasing the price of a good always changes demand by $(-1,-1,0)$, $(0,0,-1)$, or $\pm(1,1,-1)$. This condition on comparative statics is different from ones such as substitutability or complementarity; it intuitively requires that the first two goods be complementary, and that the bundle consisting of both of them be substitutable for the third good.

The connection between demand types and the comparative statics of demand is much more general than the cases addressed by Theorem 1. However, when more than one unit of each good can be demanded, the defining vectors of a demand type no longer give the possible changes in demand. Nevertheless, the main results of this paper show that a demand type can be equivalently defined as the class of valuations for which each change in prices induces a change in demand that can be built from the defining vectors of the demand type in a way that satisfies natural revealed preference conditions (Theorems 2 and 3).

In the general case, we show that one must consider changes in demand that occur as the prices of several goods change simultaneously. However, it suffices to consider changes in the price of one good at a time for demand types under which if we consider any specific pair of goods, the pair is either consistently substitutable, or consistently complementary, at a one-for-one rate (Theorem 2). These "consistent demand types" turn out to include every unimodular demand types that contains the class of all additive valuations (Proposition 1). Hence, every demand type that contains all additive valuations, and forms a domain for equilibrium existence, can be defined by conditions on how increasing the price of any one good can affect demand.

When goods are divisible, classical consumer theory shows that standard classes

[2] The conclusions of these two-good examples can also be seen as consequences of Proposition 4.13 in Baldwin and Klemperer (2014), and Proposition 3.6 in BK19, respectively.

of preferences such as substitutes and complements can be defined by conditions on the derivatives of demand—i.e., by the responses of demand to infinitesimal price changes—or, equivalently, based on the comparative statics of demand—i.e., by the responses of demand to arbitrary price changes. With indivisible goods, changes in demand are discrete—precluding the use of the former approach; nevertheless, this paper shows that the latter approach leads to simple definitions of domains of preferences that play a key role in the existence of equilibrium.

We proceed as follows. Section 2 reviews BK19's definition of demand types. Section 3 discusses the case in which at most one unit of each good can be demanded. Section 4 describes the case of consistent demand types. Section 5 considers the general case. Section 6 offers extensions of the main results. Section 7 is a conclusion. Appendix A discusses the law of demand for indivisible goods. Appendix B presents the proofs. Appendix C contains additional examples. Appendix D provides an alternative definition of consistency.

2 BK19's Definition of Demand Types

The setting follows BK19. There is a finite set I of indivisible goods. Throughout this paper, we fix a *valuation* $V : X \to \mathbb{R}$, where $X \subseteq \mathbb{Z}^I$ is a finite set of integer vectors. The *demand* at price vector $\mathbf{p} \in \mathbb{R}^I$ is

$$D(\mathbf{p}) = \arg\max_{\mathbf{x} \in X} \{V(\mathbf{x}) - \mathbf{p} \cdot \mathbf{x}\}.$$

To define demand types, BK19 studied the (tropical) geometry of the set of price vectors at which demand is nonunique, which they call the "locus of indifference prices" (LIP). We provide a self-contained summary of the key parts of their analysis. BK19 defined a *LIP facet* to be a set $F \subseteq \mathbb{R}^I$ of price vectors that lies within exactly one hyperplane, for which there exist bundles $\mathbf{x} \neq \mathbf{x}' \in X$ such that $F = \{\mathbf{p} \in \mathbb{R}^I \mid \mathbf{x}, \mathbf{x}' \in D(\mathbf{p})\}$.[3] Each LIP facet has a well-defined normal direction (given by the normal to the hyperplane within which the facet lies), which BK19 showed gives the direction in which demand changes for movements in prices that cross that LIP facet and no other LIP facet.[4]

[3]See Definition 2.2(2) in BK19. That definition formulates the condition that F lie within exactly one hyperplane as the requirement that F have "natural dimension" $n-1$ (see Footnote 7 in BK19).

[4]See Proposition 2.4(2) in BK19.

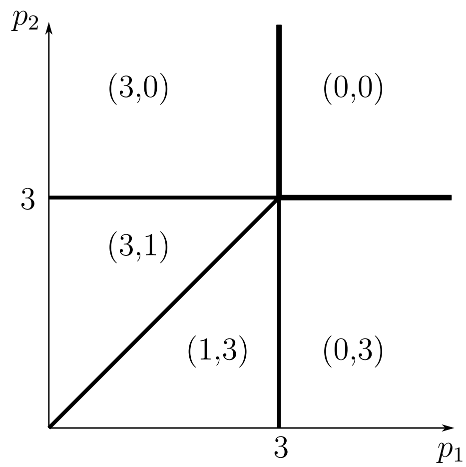

Figure 1: Demand in Example 1. The labels indicate demand in the regions of price vectors at which demand is unique. The lines represent the price vectors at which demand is not unique. The LIP facets are the five rays emanating from $(3,3)$: two horizontal, two vertical, and one (down-left) diagonal. Thus, the directions of the normals to LIP facets are $(0,1)$ for the horizontal rays, $(1,0)$ for the vertical rays, and $(1,-1)$ for the diagonal ray. Hence, the valuation is of demand type $\pm\{(1,0),(0,1),(1,-1)\}$.

In particular, the normal direction to each LIP facet must contain an integer vector. Since only the directions are relevant, we can focus on integer vectors that are *primitive* in the sense that the greatest common divisors of their components are 1.

BK19 defined demand types based on these normal primitive integer vectors. Let $\mathcal{D}$ be a set of primitive integer vectors such that if $\mathbf{d} \in \mathcal{D}$, then $-\mathbf{d} \in \mathcal{D}$.

Definition 1 (Definition 3.1 in BK19)**.** Valuation V is *of demand type* $\mathcal{D}$ if each LIP facet has a normal that is parallel to an element of $\mathcal{D}$.

The following example illustrates the mathematics of LIP facets and BK19's definition of demand types.

Example 1. There are two goods ($I = \{1,2\}$). Let

$$X = \{0,1,2,3\}^2 \setminus \{(2,3),(3,2),(3,3)\}.$$

Consider the valuation $V : X \to \mathbb{R}$ defined by $V(\mathbf{x}) = 3x_1 + 3x_2$. Figure 1 depicts demand, and illustrates that the vectors normal to LIP facets are in the directions of $(1,0)$, $(0,1)$, and $(1,-1)$. Thus, V is of demand type $\mathcal{D}$ if and only if

$\{(1,0),(0,1),(1,-1)\} \subseteq \mathcal{D}$.

There is a close connection between demand types and the existence of competitive equilibrium. Call $\mathcal{D}$ *unimodular* if for each linearly independent subset $S \subseteq \mathcal{D}$, there exists $S' \subseteq \mathbb{Z}^I$ such that $|S \cup S'| = |I|$ and the matrix whose columns are the elements of $S \cup S'$ has determinant ± 1.[5] BK19 showed that competitive equilibrium exists in all economies in which all agents have discrete-concave valuations of demand type $\mathcal{D}$ if and only if $\mathcal{D}$ is unimodular.[6] Here, letting $\operatorname{Conv} Y$ denote the convex hull of a set $Y \subseteq \mathbb{R}^I$, valuation V is *discrete-concave* if each bundle in $\operatorname{Conv} X \cap \mathbb{Z}^I$ is demanded at some price vector.

3 The Case of Binary Valuations

To illustrate the connection between demand types and comparative statics of demand developed in this paper, this section focuses on the case in which at most one unit of each good can be demanded. Our first theorem shows that in this case, a valuation is of a demand type if and only if increasing the price of one good always changes demand by a defining vector of the demand type.

Theorem 1. *Suppose that $X \subseteq \{0,1\}^I$. Valuation V is of demand type $\mathcal{D}$ if and only if, for all goods i, price vectors $\mathbf{p}$, and new prices $p'_i > p_i$ such that $D(\mathbf{p}) = \{\mathbf{x}\}$ and $D(p'_i, \mathbf{p}_{I \smallsetminus \{i\}}) = \{\mathbf{x}'\}$, the difference $\mathbf{x}' - \mathbf{x}$ is either $\mathbf{0}$ or an element of $\mathcal{D}$.*

To prove the "if" direction of Theorem 1, we simply note that the change in demand entailed by a facet normal must be induced by a change in the price of some good—and therefore, by hypothesis, be an element of $\mathcal{D}$. To prove the "only if" direction, recall that the law of demand from consumer theory states that demand is "downward sloping:" that is, that increasing the price of one good in a way that changes demand must always strictly lower demand for that good.

Lemma 1 (Law of Demand)**.** *Let $\mathbf{p}$ be a price vector, and let $p'_i > p_i$ be a new price for a good i. Suppose that $D(\mathbf{p}) = \{\mathbf{x}\}$ and that $D(p'_i, \mathbf{p}_{I \smallsetminus \{i\}}) = \{\mathbf{x}'\}$. If $\mathbf{x}' \neq \mathbf{x}$, then $x'_i < x_i$.*[7]

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[5] When $\mathcal{D}$ spans $\mathbb{R}^I$, unimodularity just requires that each nonsingular square matrix whose columns are elements of $\mathcal{D}$ have determinant ± 1.

[6] See Corollary 4.4 in BK19.

[7] Chambers and Echenique (2017) showed a similar result that considers prices at which demand

So if the price of good i is increased continuously starting at $\mathbf{p}$, demand for good i must strictly fall whenever there is any change in demand. But since at most one unit of i can be demanded, demand for i can only fall once, so demand can only change once—say at a price vector $\hat{\mathbf{p}}$.[8] The overall change in demand is then given by the change in demand entailed by the LIP facet normal for a LIP facet that contains $\hat{\mathbf{p}}$.

Note that not all elements of $\mathcal{D}$ can be obtained as changes in demand when the price of i is increased: indeed, by the law of demand, only elements of $\mathcal{D}$ that prescribe strict decreases in the demand for i can arise. Formally, given a good i, let

$$\mathcal{D}_i^- = \{\mathbf{d} \in \mathcal{D} \mid d_i < 0\}$$

denote the set of vectors in $\mathcal{D}$ that would satisfy the law of demand for the change in demand induced by an increase in the price of good i.

Corollary 1. *Suppose that $X \subseteq \{0,1\}^I$. Valuation V is of demand type $\mathcal{D}$ if and only if, for all goods i, price vectors $\mathbf{p}$, and new prices $p_i' > p_i$ such that $D(\mathbf{p}) = \{\mathbf{x}\}$ and $D(p_i', \mathbf{p}_{I \smallsetminus \{i\}}) = \{\mathbf{x}'\}$, the difference $\mathbf{x}' - \mathbf{x}$ is either $\mathbf{0}$ or an element of $\mathcal{D}_i^-$.*

We next illustrate Corollary 1 through a version of an example from the introduction.

Example 2. There are three goods ($I = \{1,2,3\}$). Suppose that $X \subseteq \{0,1\}^I$, and let

$$\mathcal{D} = \pm\{(1,0,0),(0,1,0),(0,0,1),(1,1,0),(1,1,-1)\}.$$

By Corollary 1, valuation V is of demand type $\mathcal{D}$ if and only if, for all goods i, price vectors $\mathbf{p}$, and new prices $p_i' > p_i$ such that $D(\mathbf{p}) = \{\mathbf{x}\}$ and $D(p_i', \mathbf{p}_{I \smallsetminus \{i\}}) = \{\mathbf{x}'\}$, the difference $\mathbf{x}' - \mathbf{x}$ is one of

- $\mathbf{0}$, $(-1,0,0)$, $(-1,-1,0)$, or $(-1,-1,1)$ when $i = 1$,
- $\mathbf{0}$, $(0,-1,0)$, $(-1,-1,0)$, or $(-1,-1,1)$ when $i = 2$,
- $\mathbf{0}$, $(0,0,-1)$, or $(1,1,-1)$ when $i = 3$.

is nonunique but only obtains a weak inequality (see Lemma 4 in Chambers and Echenique (2017)). We extend Lemma 1 to consider such prices in Appendix A.

[8]This step of the argument uses the hypothesis that only one good's price changes. As we show in Appendix C, the "only if" direction of Theorem 1 does not extend to simultaneous changes in the prices of multiple goods.

This condition requires that:

- increasing the price of one of the first two goods (in a way that changes demand) either simply lower demand for that good, or make the agent stop demanding the bundle of the first two goods or substitute from that bundle to the third good; and
- increasing the price of the third good (in a way that changes demand) either simply lower demand for that good, or make the agent substitute from the good to the bundle consisting of the first two goods.

Thus, when at most one unit of each good can be demanded, being of demand type $\mathcal{D}$ intuitively requires that first two goods be (weakly) complementary, and that the bundle consisting of them be (weakly) substitutable for the third good.

While Theorem 1 suggests a close connection between demand types and the comparative statics of demand, the relationship is more complicated when more than one unit of each good can be demanded.

Example 3 (Failure of the conclusion of the "only if" direction of Theorem 1 when $X \not\subseteq \{0,1\}^I$). As in Example 1, suppose that there are two goods ($I = \{1,2\}$), let

$$X = \{0,1,2,3\}^2 \smallsetminus \{(2,3),(3,2),(3,3)\},$$

and define $V : X \to \mathbb{R}$ by $V(\mathbf{x}) = 3x_1 + 3x_2$. Letting $\mathcal{D} = \pm\{(1,0),(0,1),(1,-1)\}$, Example 1 shows that V is of demand type $\mathcal{D}$. However, considering the price vector $\mathbf{p} = (1,2)$ and increasing the price of the first good to $p'_1 = 4$, we have that $D(\mathbf{p}) = \{(3,1)\}$ and $D(p'_1, p_2) = \{(0,3)\}$, while the difference $(0,3) - (3,1) = (-3,2)$ is not an element of $\mathcal{D}$.

When more than one unit of each good can be demanded, Example 3 shows that changing the price of a good need not change demand for valuations of a demand type by a defining vector of the demand type. We therefore consider changes in demand can be built from the defining vectors in a way that satisfies natural revealed preference conditions. The remainder of the paper formalizes this idea and uses it to give new definitions of demand types.

4 The Case of Consistent Demand Types

In this section, we show how a broad class of demand types can be defined in terms of conditions on how increasing the price of one good can affect demand.

4.1 Motivating Example: Strong Substitutes

We illustrate our approach using the example of strong substitutes.

Milgrom and Strulovici (2009) defined strong substitutability by building on a standard definition of substitutability for settings with indivisible goods (Kelso and Crawford, 1982; Ausubel and Milgrom, 2002), which requires that increasing the price of one good weakly raise demand for all other goods. Formally, V is a *substitutes valuation* if for all goods i, price vectors $\mathbf{p}$, and new prices $p'_i > p_i$ such that $D(\mathbf{p}) = \{\mathbf{x}\}$ and $D(p'_i, \mathbf{p}_{I \smallsetminus \{i\}}) = \{\mathbf{x}'\}$, we have that $x'_j \geq x'_i$ for all goods $j \neq i$. Milgrom and Strulovici (2009) refined substitutability by requiring that units of goods, rather than goods, be substitutes. Formally, we say that V is a *strong substitutes valuation* if it corresponds to a substitutes valuation when each unit of each good is regarded as a separate good.

There is an important connection between strong substitutability and a monotonicity property for the total number of demanded units introduced by Hatfield and Milgrom (2005). We say that V *satisfies the law of aggregate demand* if for all goods i, price vectors $\mathbf{p}$, and new prices $p'_i > p_i$ such that $D(\mathbf{p}) = \{\mathbf{x}\}$ and $D(p'_i, \mathbf{p}_{I \smallsetminus \{i\}}) = \{\mathbf{x}'\}$, we have that

$$\sum_{j \in I} x'_j \leq \sum_{j \in I} x_j.$$

For discrete-concave valuations, strong substitutability is equivalent to the conjunction of substitutability and the law of aggregate demand.

Fact 1 (Milgrom and Strulovici, 2009; Shioura and Tamura, 2015)**.** *Valuation V is a strong substitutes valuation if and only if it is a discrete-concave substitutes valuation that satisfies the law of aggregate demand.*[9]

Demand types give an alternative viewpoint on substitutability and the law of aggregate demand. To understand this viewpoint, consider a change in demand from

[9] Milgrom and Strulovici (2009) assumed that valuations are monotone; Shioura and Tamura (2015) proved the result without that assumption.

$\mathbf{x}$ to $\mathbf{x}'$ in response to an increase in the price of good i. We show that the conditions on $\mathbf{x}' - \mathbf{x}$ entailed by substitutability and the law of aggregate demand is equivalent to the condition that $\mathbf{x}' - \mathbf{x}$ can be expressed as a nonnegative linear combination of certain integer vectors.

Lemma 2. *Let i be a good. Vectors $\mathbf{x}, \mathbf{x}' \in \mathbb{R}^I$ satisfy $x'_j \geq x_j$ for all goods $j \neq i$ and $\sum_{j \in I} x'_j \leq \sum_{j \in I} x_j$ if and only if $\mathbf{x}' - \mathbf{x}$ can be expressed as a nonnegative linear combination of the vectors $-\mathbf{e}^i$ and $\{\mathbf{e}^j - \mathbf{e}^i \mid j \in I \smallsetminus \{i\}\}$.*

Here, $\mathbf{e}^j \in \mathbb{R}^I$ denotes the elementary basis vector corresponding to a good $j \in I$. Thus, the vector $-\mathbf{e}^i$ corresponds to a decrease in demand for good i, while a vector $\mathbf{e}^j - \mathbf{e}^i$ corresponds to one-for-one substitution from good i to good j. Each of these vectors clearly satisfies the conditions entailed by substitutability and the law of aggregate demand for the change in demand in response to an increase in the price of good i; Lemma 2 states that every such change in demand is a nonnegative linear combination of these vectors. Intuitively, Lemma 2 tells us that the basic demand changes for a substitutes valuation satisfies the law of aggregate demand are[10]

$$\mathcal{D}^{\text{ss}} = \{\pm \mathbf{e}^i \mid i \in I\} \cup \{\mathbf{e}^i - \mathbf{e}^j \mid i \neq j \in I\}.$$

To be more precise about the relationship between substitutability, the law of aggregate demand, and $\mathcal{D}^{\text{ss}}$, note that the vectors in Lemma 2 are precisely the vectors in $\mathcal{D}^{\text{ss}}$ that specify strict decreases in demand for good i. Thus, Lemma 2 shows that V *is a substitutes valuation that satisfies the law of aggregate demand if and only if every increase in the price of a good i induces a change a demand that can expressed as a nonnegative linear combination of vectors in $\mathcal{D}^{\text{ss}}$, each of which would separately satisfy the law of demand for the price increase.*

Example 4. In Example 3, V is a strong substitutes valuation. And the price effect $(-3, 2)$, which is induced by an increase in the price of the second good, is a nonnegative linear combination

$$(-3, 2) = 2(-1, 1) + (-1, 0)$$

of the vectors $(-1, 1)$ and $(-1, 0)$—which are the elements of $\mathcal{D}^{\text{ss}}$ that specify strict

[10] BK19 called $\mathcal{D}^{\text{ss}}$ the set of *strong substitutes vectors.*

decreases in demand for the second good (and hence satisfy the law of demand for the price increase).

BK19 provided a different viewpoint on the connection between strong substitutability and $\mathcal{D}^{\text{ss}}$ in terms of demand types.

Fact 2 (Shioura and Tamura, 2015; BK19)**.** *If V is discrete-concave, then V is a strong substitutes valuation if and only if it is of demand type $\mathcal{D}^{\text{ss}}$.*

Combining Lemma 2 and Facts 1 and 2, we see that interpreting the vectors in $\mathcal{D}^{\text{ss}}$ as the building blocks from which changes in demand can be built leads to a definition of the demand type $\mathcal{D}^{\text{ss}}$. Our new definitions of demand types are extensions of this idea.[11]

4.2 Consistent Demand Types

We now describe a class of demand types that, like strong substitutes, can be defined in terms of conditions on the changes in demand induced by increases in the price of one good.

Definition 2. A set $\mathcal{D}$ of integer vectors is *consistent* if $\mathcal{D} \subseteq \{-1, 0, 1\}^I$, and for all goods i, j, the product $d_i d_j$ is either nonnegative for all $\mathbf{d} \in \mathcal{D}$ or nonpositive for all $\mathbf{d} \in D$.

Since the elements of $\mathcal{D}$ give basic changes in demand, the first part of Definition 2 requires that any substitution and complementarity between pairs of goods be at a one-to-one rate.[12] The second part of Definition 2 requires that if we consider any specific pair i, j of goods, i and j are either consistently complementary, or consistently substitutable, over the entire demand type.[13] Here, complementarity corresponds to the case in which the product $d_i d_j$ is nonnegative for all $\mathbf{d} \in \mathcal{D}$—i.e., d_i and d_j do not have opposite signs for any $\mathbf{d} \in \mathcal{D}$; substitutability corresponds to the case in which

[11]While Fact 2 relies on discrete-concavity, we will see that discrete-concavity is not essential to our comparative statics approach.

[12]More precisely, $\mathcal{D} \subseteq \{-1, 0, 1\}^I$ holds if and only if for all valuations of demand type $\mathcal{D}$, increasing the price of a good i always makes demand for other goods change, in magnitude, by the amount by which demand for i falls. See Appendix D for a formal statement and proof.

[13]This second part of the definition of consistency has some similarities to the "sign-consistency" condition of Candogan et al. (2015). However, their condition relates to the signs of quadratic terms in a quadratic valuation, while our condition relates to changes in demand and substitutability/complementarity between pairs of goods.

the product $d_i d_j$ is nonpositive for all $\mathbf{d} \in \mathcal{D}$—i.e., d_i and d_j do not have the same sign for any $\mathbf{d} \in \mathcal{D}$.[14] Thus, for example, the set $\mathcal{D}^{\text{ss}}$, which corresponds to one-for-one substitution between all pairs of goods, is the largest $\mathcal{D}$ satisfying the first condition of Definition 2 for which the product $d_i d_j$ is nonpositive for all $\mathbf{d} \in D$.

Our second theorem shows that like the class of strong substitutes valuations, consistent demand types can be defined by conditions on how increasing the price of one good can affect demand.

Theorem 2. *Suppose that $\mathcal{D}$ is consistent. Valuation V is of demand type $\mathcal{D}$ if and only if, for all goods i, price vectors $\mathbf{p}$, and new prices $p'_i > p_i$ such that $D(\mathbf{p}) = \{\mathbf{x}\}$ and $D(p'_i, \mathbf{p}_{I \smallsetminus \{i\}}) = \{\mathbf{x}'\}$, the difference $\mathbf{x}' - \mathbf{x}$ is a nonnegative linear combination of elements of $\mathcal{D}_i^-$.*

Theorem 2 shows that if $\mathcal{D}$ is consistent, then a valuation is of demand type $\mathcal{D}$ if and only if increasing the price of one good always induces a change in demand that can be expressed as nonnegative linear combination of members of $\mathcal{D}_i^-$—i.e., a nonlinear combination of elements of $\mathcal{D}$, each of which would separately satisfy the law of demand for the price increase. Thus, interpreting elements of $\mathcal{D}$ as the building blocks from which changes in demand in response to a change in the price of a good can be built leads to a new definition of demand types in the consistent case. Unlike Definition 1, this definition is based purely on comparative statics and does not rely on understanding the geometry of LIPs or LIP facets.

The "only if" direction of Theorem 2 follows from an argument of BK19 that they used to show that the class of all substitutes valuations corresponds to an "ordinary substitutes" demand type.[15] That argument can in fact be used to show that if a valuation is of demand type $\mathcal{D}$, then for each change in demand (between price vectors at which demand is unique), there is a sequence of intermediate price vectors and bundles demanded at these price vectors that break the overall change in demand into steps in the directions of elements of $\mathcal{D}$. Combining these steps yields the "only if" direction of Theorem 2. This approach in fact shows that for a valuation of demand

[14]More precisely, given goods i, j, the product $d_i d_j$ is nonnegative (resp. nonpositive) for all $\mathbf{d} \in \mathcal{D}$ if and only if increasing the price of i always weakly lowers (resp. weakly raises) demand for all valuations of demand type $\mathcal{D}$. See Appendix D for a formal statement and proof.

[15]This result is Proposition 3.6 in BK19. Their Proposition 3.8 uses a similar argument to show that the class of all complements valuations corresponds to an "ordinary complements" demand type.

type $\mathcal{D}$, the change in demand in response to an increase in the price of good i must be a nonnegative integer combination of elements of $\mathcal{D}_i^-$.

The proof of the "if" direction uses a different argument. We cannot apply BK19's argument as the hypothesis does not give intermediate price vectors that break the overall change in demand into steps. To show that a LIP facet normal must be parallel to an element of $\mathcal{D}$, we consider a good i whose change in demand entailed by the facet normal is as small as possible (while remaining nonzero). We apply the hypothesis to a small change in the price of good i near the LIP facet: the change $\mathbf{x}' - \mathbf{x}$ in demand entailed by the facet normal can be expressed as a nonnegative linear combination of elements of $\mathcal{D}_i^-$. Consistency entails that demand for goods other than i must change weakly less than demand for i. But since demand for good i changed the least by hypothesis, we can conclude that all goods whose demand changes must change by the same amount. In light of consistency, for any good whose demand changes at all, every element of $\mathcal{D}_i^-$ appearing in the expression of $\mathbf{x}' - \mathbf{x}$ must prescribe a change in the demand for the good. Hence, only one element of $\mathcal{D}_i^-$ can appear in the expression of $\mathbf{x}' - \mathbf{x}$, so $\mathbf{x}' - \mathbf{x}$ must be proportional to an element of $\mathcal{D}$.

In the special case of the strong substitutes demand type, Lemma 2 describes the vectors that can be expressed as nonnegative linear combinations of elements of $(\mathcal{D}^{\text{ss}})_i^-$. Hence, since $\mathcal{D}^{\text{ss}}$ is consistent, combining Theorem 2 with Lemma 2 leads to a simple characterization of the demand type $\mathcal{D}^{\text{ss}}$.

Corollary 2. *Valuation V is of demand type $\mathcal{D}^{\text{ss}}$ if and only if it is a substitutes valuation that satisfies the law of aggregate demand.*[16]

As another illustration, we next revisit the demand type described in Example 2.

Example 5. As in Example 2, suppose that there are three goods ($I = \{1, 2, 3\}$), and let

$$\mathcal{D} = \pm\{(1, 0, 0), (0, 1, 0), (0, 0, 1), (1, 1, 0), (1, 1, -1)\}.$$

Note that $\mathcal{D}$ is consistent. Hence, by Theorem 2, valuation V is of demand type $\mathcal{D}$ if and only if, for all goods i, price vectors $\mathbf{p}$, and new prices $p_i' > p_i$ such that $D(\mathbf{p}) = \{\mathbf{x}\}$ and $D(p_i', \mathbf{p}_{I \smallsetminus \{i\}}) = \{\mathbf{x}'\}$, the difference $\mathbf{x}' - \mathbf{x}$ is a nonnegative linear combination of

[16] Corollary 2 is a version of Fact 2 that applies to valuations that are not discrete-concave. Indeed, by Fact 1, the case of Corollary 2 for discrete-concave valuations is equivalent to Fact 2.

- $(-1, 0, 0)$, $(-1, -1, 0)$, and $(-1, -1, 1)$ when $i = 1$,
- $(0, -1, 0)$, $(-1, -1, 0)$, and $(-1, -1, 1)$ when $i = 2$,
- $(0, 0, -1)$ and $(1, 1, -1)$ when $i = 3$.

Note that this condition can be written, dually, as

- $x'_1 - x_1 \leq x'_2 - x_2 \leq x_3 - x'_3 \leq 0$ when $i = 1$,
- $x'_2 - x_2 \leq x'_1 - x_1 \leq x_3 - x'_3 \leq 0$ when $i = 2$,
- $0 \leq x'_1 - x_1 = x'_2 - x_2 \leq x_3 - x'_3$ when $i = 3$.

As discussed in Example 2, being of demand type $\mathcal{D}$ thus intuitively requires that the first two goods be (weakly) complementary at a one-for-one rate, and that the bundle consisting of one unit of each of them be (weakly) substitutable for the third good.

There is also a connection between consistency and unimodularity: if $\mathcal{D}$ is unimodular and contains each elementary basis vector, then $\mathcal{D}$ is consistent.

Proposition 1. *If $\mathcal{D}$ is unimodular and contains $\mathbf{e}^i$ for each good i, then $\mathcal{D}$ is consistent.*

The hypothesis that $\mathcal{D}$ contains the elementary basis vectors is fairly mild: it simply requires that changing the price of a good could affect only demand for that good—or, equivalently, that every additive valuation be of demand type $\mathcal{D}$.[17] Thus, every demand type that contains all additive valuations and is compatible with the guaranteed existence of competitive equilibrium can be defined in terms of simple conditions on the comparative statics of demand in response to an increase in the price of one good.

[17]Indeed, $\mathcal{D}$ contains the elementary basis vectors if and only if every additive valuation is of demand type $\mathcal{D}$ (see the proof of Proposition 4.13 in Baldwin and Klemperer (2014)). However, there are unimodular $\mathcal{D}$ that are not consistent (hence, in particular do not contain the elementary basis vectors)—such as $\mathcal{D} = \pm\{(1, -1), (2, -1)\}$. Nevertheless, even if $\mathcal{D}$ does not contain the elementary basis vectors, applying a suitably chosen (unimodular) basis change to $\mathbb{Z}^I$ would make the transform of $\mathcal{D}$ contain the elementary basis vectors.

5 The General Case

For inconsistent $\mathcal{D}$, considering changes in the price of one good at a time may not suffice to define demand type $\mathcal{D}$—as the following example illustrates.

Example 6 (Failure of Theorem 2 if $\mathcal{D} \not\subseteq \{-1,0,1\}^I$). There are two goods ($I = \{1,2\}$). Let $X = \{(3,0),(0,2)\}$, and define $V : X \to \mathbb{R}$ by $V(\mathbf{x}) = x_1 + x_2$.

Let $\mathcal{D} = \pm\{(1,-1),(2,-1)\} \not\subseteq \{-1,0,1\}^I$.[18] Valuation V is not of demand type $\mathcal{D}$. Indeed, the set of price vectors at which demand is nonunique is

$$L = \{\mathbf{p} \mid 3p_1 - 2p_2 = 1\}.$$

Since bundles $(3,0)$ and $(0,2)$ are both demanded at each price vector in L, the set L is a LIP facet. And the normal to L is in the direction of $(3,-2)$—which is not an element of $\mathcal{D}$.

However, the hypothesis of the "if" direction of Theorem 2 does hold. Indeed, let $\mathbf{p}$ be a price vector, let $p'_i > p_i$ be a new price for a good i, and suppose that $D(\mathbf{p}) = \{\mathbf{x}\}$ and that $D(p'_i, \mathbf{p}_{I\smallsetminus\{i\}}) = \{\mathbf{x}'\}$. If $\mathbf{x}' = \mathbf{x}$, then trivially $\mathbf{x}' - \mathbf{x}$ is a nonnegative linear combination of elements of $\mathcal{D}_i^-$. Otherwise, if the first good's price was increased (i.e., $i = 1$), then by the law of demand, we must have that $\mathbf{x} = (3,0)$ and that $\mathbf{x}' = (0,2)$. In this case, as

$$\mathcal{D}_1^- = \{(-1,1),(-2,1)\} \quad \text{and} \quad (-3,2) = (-1,1) + (-2,1),$$

the difference $\mathbf{x}' - \mathbf{x}$ is a nonnegative linear combination of elements of $\mathcal{D}_i^-$. Similarly, if the second good's price was increased (i.e., $i = 2$), then by the law of demand, we must have that $\mathbf{x} = (0,2)$ and that $\mathbf{x}' = (3,0)$. In this case, as

$$\mathcal{D}_2^- = \{(1,-1),(2,-1)\} \quad \text{and} \quad (3,-2) = (1,-1) + (2,-1),$$

the difference $\mathbf{x}' - \mathbf{x}$ is again a nonnegative linear combination of elements of $\mathcal{D}_i^-$.

The issue in Example 6 is that three-for-two substitution between goods can masquerade as a combination of two-for-one and one-for-one substitution when prices are changed one at a time. As we show in Appendix C, a similar issue can arise if $\mathcal{D} \subseteq \{-1,0,1\}^I$ but the second condition of Definition 2 is not satisfied.

[18] As discussed in Footnote 17, this $\mathcal{D}$ is unimodular but not consistent.

Hence, we need to consider simultaneous changes in the prices of multiple goods to define inconsistent demand types in terms of comparative statics of demand. For such price changes, recall that law of demand states that if a change in prices induces a change in demand, then the value of the change in demand must be negative when evaluated with respect to the change in prices.

Lemma 1′ (Law of Demand)**.** *Let* $\mathbf{p}, \mathbf{p}'$ *be price vectors. Suppose that* $D(\mathbf{p}) = \{\mathbf{x}\}$ *and that* $D(\mathbf{p}') = \{\mathbf{x}'\}$*. If* $\mathbf{x}' \neq \mathbf{x}$, *then* $(\mathbf{p}' - \mathbf{p}) \cdot (\mathbf{x}' - \mathbf{x}) < 0$.[19]

We write

$$\mathcal{D}(\mathbf{p}, \mathbf{p}') = \{\mathbf{d} \in \mathcal{D} \mid (\mathbf{p}' - \mathbf{p}) \cdot \mathbf{d} < 0\}$$

for the set of elements of $\mathcal{D}$ that would satisfy the law of demand for a change in prices from $\mathbf{p}$ to $\mathbf{p}'$. To compare with the case in which only one good's price changes, note that $\mathcal{D}_i^- = \mathcal{D}\left(\mathbf{p}, (p_i', \mathbf{p}_{I \smallsetminus \{i\}})\right)$ for all goods i, price vectors $\mathbf{p}$, and new prices $p_i' > p_i$.

Our third theorem shows that when $\mathcal{D}$ is finite, a valuation is of demand type $\mathcal{D}$ if and only if changing the price vector from $\mathbf{p}$ to $\mathbf{p}'$ induces a change in demand that is a nonnegative linear combination of elements of $\mathcal{D}(\mathbf{p}, \mathbf{p}')$.

Theorem 3. *Suppose that* $\mathcal{D}$ *is finite. Valuation* V *is of demand type* $\mathcal{D}$ *if and only if, for all price vectors* $\mathbf{p}, \mathbf{p}'$ *such that* $D(\mathbf{p}) = \{\mathbf{x}\}$ *and* $D(\mathbf{p}') = \{\mathbf{x}'\}$, *the difference* $\mathbf{x}' - \mathbf{x}$ *is a nonnegative linear combination of elements of* $\mathcal{D}(\mathbf{p}, \mathbf{p}')$.

Theorem 3 shows that by interpreting the elements of $\mathcal{D}$ as the building blocks from which changes in demand in response to simultaneous changes in the prices of several goods can be built, we obtain a new definition of demand types even beyond the consistent case. Unlike Definition 1, this definition does not rely on understanding the geometry of LIPs.

Like the "only if" direction of Theorem 2, the "only if" direction of Theorem 3 by applying an argument of BK19.[20] For the "if" direction, to understand why considering simultaneous changes in the prices of several goods leads to a definition of inconsistent demand types in terms of comparative statics, let us revisit Example 6.

[19] Lemma 1′ is a version of Proposition 3.E.4 in Mas-Colell et al. (1995) for settings with indivisible goods. Chambers and Echenique (2017) showed a similar result that considers prices at which demand is nonunique but only obtains a weak inequality (see Lemma 4 in Chambers and Echenique (2017)). We extend Lemma 1′ to consider such prices in Appendix A.

[20] Similar to the "only if" direction of Theorem 2, this approach in fact shows for a valuation of demand type $\mathcal{D}$, the change in demand in response to a change in prices from $\mathbf{p}$ to $\mathbf{p}'$ must be a nonnegative integer combination of elements of $\mathcal{D}(\mathbf{p}, \mathbf{p}')$.

In that example, we can distinguish three-for-two substitution from combinations of one-for-one and two-for-one substitution by considering a price increase for which the price of the second good increases by slightly more than $\frac{3}{2}$ times the increase in the price of the first good. Indeed, for such a price increase, one-for-one substitution would entail substitution from one unit of the second good to one unit of the first good and two-for-one substitution would entail substitution from two units of the first good to one unit of the second good—neither of which would increase the total number of demanded units. On the other hand, three-for-two substitution entails substitution from two units of the second good to three units of the first good—a strict increase in the total number of demanded units.

Example 7. In the setting of Example 6, let $0 < \epsilon < \frac{1}{4}$, and consider the price vectors $\mathbf{p} = (1 + \epsilon, 0)$ and $\mathbf{p}' = (3 - \epsilon, 3)$. Since $\mathbf{p}' - \mathbf{p} = (2 - 2\epsilon, 3)$, we have that

$$\mathcal{D}(\mathbf{p}, \mathbf{p}') = \{(1, -1), (-2, 1)\}.$$

Demand at $\mathbf{p}$ and $\mathbf{p}'$ is given by $D(\mathbf{p}) = (0, 2)$ and $D(\mathbf{p}') = (3, 0)$. The change in demand $(3, 0) - (0, 2) = (3, -2)$ is not a nonnegative linear combination of elements of $\mathcal{D}(\mathbf{p}, \mathbf{p}')$: indeed, we have that $d_1 + d_2 \leq 0$ for all $\mathbf{d} \in \mathcal{D}(\mathbf{p}, \mathbf{p}')$, but that $3 - 2 = 1 > 0$.

The proof of "only if" direction of Theorem 3 is based on a similar strategy. In Example 7, the change in prices from $\mathbf{p}$ to $\mathbf{p}'$ nearly lies on on the LIP facet L (from Example 6) along which both $(3, 0)$ and $(0, 2)$ are demanded. For the general argument, if there is a LIP facet that is not normal to any element of $\mathcal{D}$, then we consider a change in prices from $\mathbf{p}$ to $\mathbf{p}'$ that (nearly) lies on the LIP facet such that $\mathbf{p}' - \mathbf{p}$ is not (nearly) normal to any element of $\mathcal{D}$. We complete the argument by showing that the induced change in demand cannot be a nonnegative linear combination of elements of $\mathcal{D}(\mathbf{p}, \mathbf{p}')$.

6 Extensions

This section offers extensions of the main results to the case in which $\mathcal{D}$ is infinite, and to comparative statics between price vectors at which demand is nonunique.

6.1 Allowing for Infinite $\mathcal{D}$

As we show in Appendix C, Theorem 3 may not hold if $\mathcal{D}$ is infinite.[21] But, as we present here, a simple extension allows for infinite $\mathcal{D}$. In effect, the conclusion of Theorem 3 holds if one restricts to elements of $\mathcal{D}$ that lie in a sufficiently large box.

Corollary 3. *Valuation V is of demand type $\mathcal{D}$ if and only if there exists M such that letting $\widehat{\mathcal{D}} = \mathcal{D} \cap [-M, M]^I$, for all price vectors $\mathbf{p}, \mathbf{p}'$ such that $D(\mathbf{p}) = \{\mathbf{x}\}$ and $D(\mathbf{p}') = \{\mathbf{x}'\}$, the difference $\mathbf{x}' - \mathbf{x}$ is a nonnegative linear combination of elements of $\widehat{\mathcal{D}}(\mathbf{p}, \mathbf{p}')$.*

6.2 Comparative Statics When Demand is Nonunique

Sections 3–5 focused on changes in demand between price vectors at which demand is unique. We now investigate how demand types constrain changes in demand between other price vectors. Specifically, we show that if a valuation is of demand type $\mathcal{D}$, then for all price vectors $\mathbf{p}, \mathbf{p}'$ and all bundles $\mathbf{x} \in D(\mathbf{p})$, there exists a (potentially non-integer) bundle $\mathbf{x}' \in \operatorname{Conv} D(\mathbf{p}')$ satisfying similar conditions to Theorem 3 and Corollary 3.

Proposition 2. *If V is of demand type $\mathcal{D}$, then for all price vectors $\mathbf{p}, \mathbf{p}'$ and all $\mathbf{x} \in D(\mathbf{p})$, there exists $\mathbf{x}' \in \operatorname{Conv} D(\mathbf{p}')$ such that $\mathbf{x}' - \mathbf{x}$ is a nonnegative linear combination of elements of $\mathcal{D}(\mathbf{p}, \mathbf{p}')$.*

Proposition 2 shows that the approach of defining demand types based on conditions on the comparative statics of demand naturally extends to price vectors at which demand is nonunique if one considers the convex hulls of demand sets.

To prove Proposition 2, we express $\mathbf{x}$ as a convex combination of bundles that are uniquely demanded at price vectors near $\mathbf{p}$. We then consider demand at corresponding price vectors near $\mathbf{p}'$ and construct a element of $\operatorname{Conv} D(\mathbf{p}')$ based on a convex combination analogous to $\mathbf{x}$. Applying the "only if" direction of Corollary 3 to nearby price vectors then shows that $\mathbf{x}' - \mathbf{x}$ must be a nonnegative linear combination of elements of $\mathcal{D}(\mathbf{p}, \mathbf{p}')$.

[21] One important example of an infinite $\mathcal{D}$ is the $\mathcal{D}$ that defines the *ordinary substitutes demand type*, which corresponds to the class of all substitutes valuations (see Definition 3.5 and Proposition 3.6 in BK19). However, it turns out that the conclusion of Theorems 2 and 3 do hold for this $\mathcal{D}$: this conclusion follows from the characterization of the ordinary substitutes demand type as the class of all substitutes valuations.

Note that in general, the conclusion of Proposition 2 would not hold if $\mathbf{x}'$ were required to be integer or to be in $D(\mathbf{p}')$ (see Example 6 in Danilov et al. (2003), as well as Appendix C). But when $\mathcal{D}$ is unimodular and V discrete-concave, it turns out that $\mathbf{x}'$ can be chosen to be in $D(\mathbf{p}')$; this result leads to a characterization of the discrete-concave valuations of unimodular demand types that explicitly considers price vectors at which demand is nonunique.

Corollary 4. *Suppose that $\mathcal{D}$ is unimodular and that V is discrete-concave. Valuation V is of demand type $\mathcal{D}$ if and only if, for all price vectors $\mathbf{p}, \mathbf{p}'$ and all $\mathbf{x} \in D(\mathbf{p})$, there exists $\mathbf{x}' \in D(\mathbf{p}')$ such that $\mathbf{x}' - \mathbf{x}$ is a nonnegative linear combination of elements of $\mathcal{D}(\mathbf{p}, \mathbf{p}')$.*

The "if" direction of Corollary 4 follows from Theorem 3 as unimodular $\mathcal{D}$ are finite. To derive the "only if" direction of Corollary 4 from Proposition 2, we use the existence of competitive equilibrium in an auxiliary economy.

7 Conclusion

This paper shows how the classification of valuations over indivisible goods into "demand types," which BK19 introduced via a tropical geometric approach, can be defined more conventionally in terms of simple conditions on the comparative statics of demand. Indeed, we show that interpreting the defining vectors of a demand type as the building blocks from which changes in demand can be built leads to a new definition of demand types. For "consistent" demand types, which include every demand type that contains all additive valuations and is compatible with the guaranteed existence of competitive equilibrium, it suffices to consider changes in the price of one good at a time; for general demand types, we must consider simultaneous changes in the prices of several goods. These results show that even though indivisibilities preclude analyzing demand using infinitesimal changes in prices, the classical consumer theory approach of studying preferences in terms of the comparative statics of demand still works well with indivisible goods.

A The Law of Demand when Demand is Nonunique

In this appendix, we derive a versions of the law of demand that apply for price vectors at which demand is nonunique. Our result strengthens Lemmata 1 and 1′.

Lemma A.1. *Let $\mathbf{p}, \mathbf{p}'$ be price vectors, and let $\mathbf{x} \in D(\mathbf{p})$ and $\mathbf{x}' \in D(\mathbf{p}')$.*

(a) We have that $(\mathbf{p}' - \mathbf{p}) \cdot (\mathbf{x}' - \mathbf{x}) \leq 0$, with equality if and only if $\mathbf{x} \in D(\mathbf{p}')$ and $\mathbf{x}' \in D(\mathbf{p})$.

(b) If $D(\mathbf{p}) = \{\mathbf{x}\}$ and $\mathbf{x}' \neq \mathbf{x}$, then $(\mathbf{p}' - \mathbf{p}) \cdot (\mathbf{x}' - \mathbf{x}) < 0$.

The weak inequality of Lemma A.1(a) is part of Lemma 4 in Chambers and Echenique (2017). The proof of the full result also follows from the proof of Lemma 4 in Chambers and Echenique (2017).

Proof. We first prove Part (a). Since $\mathbf{x} \in D(\mathbf{p})$, we have that

$$\begin{aligned} V(\mathbf{x}) - \mathbf{p} \cdot \mathbf{x} &\geq V(\mathbf{x}') - \mathbf{p} \cdot \mathbf{x}' \\ V(\mathbf{x}) - V(\mathbf{x}') &\geq \mathbf{p} \cdot (\mathbf{x} - \mathbf{x}'), \end{aligned} \tag{A.1}$$

with equality if and only if $\mathbf{x}' \in D(\mathbf{p})$. Similarly, we have that

$$V(\mathbf{x}') - V(\mathbf{x}) \geq \mathbf{p}' \cdot (\mathbf{x}' - \mathbf{x}), \tag{A.2}$$

with equality if and only if $\mathbf{x} \in D(\mathbf{p})$. Adding (A.1) and (A.2), we have that

$$0 \geq \mathbf{p}' \cdot (\mathbf{x}' - \mathbf{x}) + \mathbf{p} \cdot (\mathbf{x} - \mathbf{x}') = (\mathbf{p}' - \mathbf{p}) \cdot (\mathbf{x}' - \mathbf{x}),$$

with equality if and only if $\mathbf{x}' \in D(\mathbf{p})$ and $\mathbf{x} \in D(\mathbf{p})$—as desired.

To prove Part (b), note that if $D(\mathbf{p}) = \{\mathbf{x}\}$ and $\mathbf{x}' \neq \mathbf{x}$, then $\mathbf{x}' \notin D(\mathbf{p})$. Hence, Part (b) follows from Part (a). □

B Proofs

Several proofs use the following technical characterization of demand types, which is "dual" to Definition 1 and follows from Proposition 2.20 in BK19.

Fact B.1. *Valuation V is of demand type $\mathcal{D}$ if and only if, for all price vectors $\mathbf{p}$ for which* $\operatorname{Conv} D(\mathbf{p})$ *is a line segment,* $\operatorname{Conv} D(\mathbf{p})$ *is parallel to an element of $\mathcal{D}$.*

Several proofs also use the following technical claim to connect Fact B.1 to properties of the comparative statics of demand.

Claim B.1. *If $\hat{\mathbf{p}}$ is a price vector such that* $\operatorname{Conv} D(\hat{\mathbf{p}})$ *is a line segment with endpoints $\mathbf{x}, \mathbf{x}'$, then for each vector $\mathbf{s} \in \mathbb{R}^I$ with $\mathbf{s} \cdot (\mathbf{x}' - \mathbf{x}) < 0$, there exists $\lambda > 0$ such that $D(\hat{\mathbf{p}} - \lambda\mathbf{s}) = \{\mathbf{x}\}$ and $D(\hat{\mathbf{p}} + \lambda\mathbf{s}) = \{\mathbf{x}'\}$.*

Proof. Let $\mathbf{x}' - \mathbf{x} = n\mathbf{g}$, where $\mathbf{g}$ is a primitive integer vector, and $n > 0$. Without loss of generality, we can assume that $\|\mathbf{s}\| = 1$. By the upper hemicontinuity of demand, there exists λ such that $D(\mathbf{p}) \subseteq D(\hat{\mathbf{p}})$ for all $\mathbf{p}$ with $\|\mathbf{p} - \hat{\mathbf{p}}\| \leq \lambda$.

If $\mathbf{x}'' = \mathbf{x} + \ell\mathbf{g} \in D(\hat{\mathbf{p}} - \lambda\mathbf{s})$, then since $\mathbf{x} \in D(\hat{\mathbf{p}})$, applying Lemma A.1(a) to the price change from $\hat{\mathbf{p}} - \lambda\mathbf{s}$ to $\hat{\mathbf{p}}$, we see that

$$0 \geq \lambda\mathbf{s} \cdot (\mathbf{x}'' - \mathbf{x}) = \ell\lambda\mathbf{s} \cdot \mathbf{g} = \ell\lambda\frac{\mathbf{s} \cdot (\mathbf{x}' - \mathbf{x})}{n},$$

so $\ell \leq 0$ must hold. As $D(\mathbf{p}) \subseteq D(\hat{\mathbf{p}}) \subseteq \{\mathbf{x} + \ell\mathbf{g} \mid 0 \leq \ell \leq n\}$ holds by construction, we must have that $D(\mathbf{p}) = \{\mathbf{x}\}$. A similar argument shows that $D(\hat{\mathbf{p}}+\lambda\mathbf{s}) = \{\mathbf{x}'\}$. □

B.1 Proof of Theorem 1

We first prove the "if" direction. The argument uses the "if" direction of Fact B.1 to show that V must be of demand type $\mathcal{D}$. Let $\hat{\mathbf{p}}$ be a price vector such that $\operatorname{Conv} D(\hat{\mathbf{p}})$ is a line segment, say with endpoints $\mathbf{x}$ and $\mathbf{x}'$. Without loss of generality, we can assume that there exists a good i such that $x_i > x'_i$. By Claim B.1 applied to the vector $\mathbf{s} = \mathbf{e}^i$, there exists $\lambda > 0$ such that letting $\mathbf{p} = (\hat{p}_i - \lambda, \hat{\mathbf{p}}_{I \smallsetminus \{i\}})$ and $p'_i = \hat{p}_i + \lambda$, we have that $D(\mathbf{p}) = \{\mathbf{x}\}$ and that $D(p'_i, \mathbf{p}_{I \smallsetminus \{i\}}) = \{\mathbf{x}'\}$. The hypothesis of the "if" direction then entails that $\mathbf{x}' - \mathbf{x} \in \mathcal{D}$. In particular, $\operatorname{Conv} D(\hat{\mathbf{p}})$ is parallel to an element of $\mathcal{D}$. Since $\hat{\mathbf{p}}$ was arbitrary, the "if" direction of the proposition thus follows from the "if" direction of Fact B.1.

We next prove the "only if" direction. Let $\mathbf{p}$ be a price vector, let $p'_i > p_i$ be a new price for a good i, and suppose that $D(\mathbf{p}) = \{\mathbf{x}\}$ and that $D(p'_i, \mathbf{p}_{I \smallsetminus \{i\}}) = \{\mathbf{x}'\}$, where $\mathbf{x}' \neq \mathbf{x}$. Since $X \subseteq \{0, 1\}^I$, we must have that $x_i = 1$ and that $x'_i = 0$. Let

$$\hat{p}_i = \max\left\{p''_i \leq p'_i \,\middle|\, \mathbf{x} \in D(p''_i, \mathbf{p}_{I \smallsetminus \{i\}})\right\}; \tag{B.1}$$

the maximum exists due to the upper hemicontinuity of demand. Applying Lemma A.1(b) to the price change from $\mathbf{p}$ to $(\hat{p}_i, \mathbf{p}_{I \smallsetminus \{i\}})$, we see that $\hat{x}_i < x_i = 1$ for all $\hat{\mathbf{x}} \in D(\mathbf{p}) \smallsetminus \{\mathbf{x}\}$. Similarly, applying Lemma A.1(b) to the price change from $(p'_i, \mathbf{p}_{I \smallsetminus \{i\}})$ to $(\hat{p}_i, \mathbf{p}_{I \smallsetminus \{i\}})$, we see that $\hat{x}_i > x'_i = 0$ for all $\hat{\mathbf{x}} \in D(\mathbf{p}) \smallsetminus \{\mathbf{x}'\}$. It follows that $D(\hat{p}_i, \mathbf{p}_{I \smallsetminus \{i\}}) \subseteq \{\mathbf{x}, \mathbf{x}'\}$. But since demand is upper hemicontinuous, (B.1) implies that $D(\hat{p}_i, \mathbf{p}_{I \smallsetminus \{i\}}) \neq \{\mathbf{x}\}$. Therefore, we must have that $D(\hat{p}_i, \mathbf{p}_{I \smallsetminus \{i\}}) = \{\mathbf{x}, \mathbf{x}'\}$. The "only if" direction of Fact B.1 then implies that the difference $\mathbf{x}' - \mathbf{x}$ must be proportional to an element of $\mathcal{D}$. As $\mathbf{x}' - \mathbf{x}$ is a nonzero element of $\{-1, 0, 1\}^I$, it must be a primitive integer vector, and hence in fact an element of $\mathcal{D}$.

B.2 Proof of Lemma 1

Lemma 1 is the special case of Lemma A.1(b) in which $\mathbf{p}' = (p'_i, \mathbf{p}_{I \smallsetminus \{i\}})$ and $D(\mathbf{p}') = \{\mathbf{x}'\}$.

B.3 Proof of Lemma 2

We first prove the "if" direction. Suppose that

$$\mathbf{x}' - \mathbf{x} = -\alpha_i \mathbf{e}^i + \sum_{j \in I \smallsetminus \{i\}} \alpha_j (\mathbf{e}^j - \mathbf{e}^i),$$

where $\alpha_j \geq 0$ for $j \in I$. We then have that $x'_j - x_j = \alpha_j \geq 0$ for all goods $j \neq i$, and that

$$\sum_{j \in I} x'_j - \sum_{j \in I} x_j = -\alpha_i \leq 0.$$

We next prove the "only if" direction. Let

$$\alpha_i = \sum_{j \in I} x_j - \sum_{j \in I} x'_j \geq 0,$$

and, for goods $j \neq i$, let $\alpha_j = x'_j - x_j \geq 0$. We then have that

$$x'_i - x_i = \left(\sum_{j \in I} x'_j - \sum_{j \in I} x_j \right) - \left(\sum_{j \in I \smallsetminus \{i\}} (x'_j - x_j) \right) = -\alpha_i - \sum_{j \in I \smallsetminus \{i\}} \alpha_j.$$

It follows that

$$\mathbf{x}' - \mathbf{x} = -\alpha_i \mathbf{e}^i + \sum_{j \in I \smallsetminus \{i\}} \alpha_j (\mathbf{e}^j - \mathbf{e}^i)$$

—expressing $\mathbf{x}' - \mathbf{x}$ as a nonnegative linear combination $-\mathbf{e}^i$ and $\{\mathbf{e}^j - \mathbf{e}^i \mid j \in I \smallsetminus \{i\}\}$.

B.4 Proof of Theorem 2

Proof of the "only if" direction. The proof of the "only if" direction relies on the following lemma, which we also use in the proof of Theorem 3.

Lemma B.1. *If V is of demand type $\mathcal{D}$, then for all price changes $\mathbf{t}$, there exists an open, dense set S of price vectors such that for all $\mathbf{p}^0 \in S$ with $D(\mathbf{p}^0 + \mathbf{t}) \neq D(\mathbf{p}^0)$, there are price vectors $\mathbf{p}^1, \ldots, \mathbf{p}^k = \mathbf{p}^0 + \mathbf{t}$ and bundles $\mathbf{x}^\ell \in D(\mathbf{p}^\ell)$ such that for $1 \leq \ell \leq k$, the difference $\mathbf{x}^\ell - \mathbf{x}^{\ell-1}$ is proportional to an element of $\mathcal{D}$ and satisfies $\mathbf{t} \cdot (\mathbf{x}^\ell - \mathbf{x}^{\ell-1}) < 0$.*[22]

The proof of Lemma B.1 follows BK19's argument to prove their Proposition 3.6.

Proof. It follows from Proposition 2.20 in BK19 that the LIP

$$L = \{\mathbf{p} \in \mathbb{R}^I \bigm| |D(\mathbf{p})| > 1\}$$

is contained in a finite union $H_1 \cup H_2 \cup \cdots \cup H_m$ of hyperplanes in $\mathbb{R}^I$. Consider the set

$$W = \{\mathbf{p} \in L \mid \text{Conv}\, D(\mathbf{p}) \text{ is not a line segment}\}$$

of price vectors. It follows from Proposition 2.20 in BK19 that W is lies in a finite union of $(|I| - 2)$-dimensional planes in $\mathbb{R}^I$. In particular, the set $W' = W + \mathbb{R}\mathbf{t}$ is closed and is contained in a finite union $H'_1 \cup H'_2 \cup \cdots \cup H'_{m'}$ of hyperplanes in $\mathbb{R}^I$.

Now let

$$S = \mathbb{R}^I \smallsetminus \big((H_1 \cup \cdots \cup H_m) \cup ((H_1 - \mathbf{t}) \cup \cdots \cup (H_m - \mathbf{t})) \cup (H'_1 \cup \cdots \cup H'_{m'}) \big),$$

which is open and dense by construction. Note that $S \subseteq \mathbb{R}^I \smallsetminus \big(L \cup (L - \mathbf{t}) \cup W'\big)$.

[22] The (1) $\Longrightarrow$ (2) implication of Theorem 4.4 in Section 4.2 of Baldwin and Klemperer's 2014 working paper shows that a similar conclusion applies for all $\mathbf{p}^0$ for which $|D(\mathbf{p}^0)| = 1$, but Lemma B.1 is sufficient for our purposes.

Since $\mathbf{p}^0 \notin H_1 \cup H_2 \cup \cdots \cup H_m$, the line $\mathbf{p}^0 + \mathbb{R}\mathbf{t}$ can only meet $H_1 \cup H_2 \cup \cdots \cup H_m$ at finitely many points. In particular, the line segment with endpoints $\mathbf{p}^0$ and $\mathbf{p}^0 + \mathbf{t}$ can only meet L at finitely many points; call the points of intersection $\hat{\mathbf{p}}^1, \hat{\mathbf{p}}^2, \ldots, \hat{\mathbf{p}}^{k-1}$, and suppose they arise in that order as the segment is traversed from $\mathbf{p}^0$ to $\mathbf{p}^0 + \mathbf{t}$. Since $\mathbf{p}^0 \notin H'_1 \cup H'_2 \cup \cdots \cup H'_{m'}$, none of the points of intersection can be in W.

Suppose that $D(\mathbf{p}^0 + \mathbf{t}) \neq D(\mathbf{p}^0)$. In this case, due to the upper hemicontinuity of demand, we must have that $k \geq 2$. Let $\mathbf{p}^\ell = \frac{\hat{\mathbf{p}}^\ell + \hat{\mathbf{p}}^{\ell+1}}{2}$ for $1 \leq \ell \leq k-1$, and let $\mathbf{p}^k = \mathbf{p}^0 + \mathbf{t}$. By construction, we have that $\mathbf{p}^\ell \notin L$ for all $0 \leq \ell \leq k$. Hence, we that $|D(\mathbf{p}^\ell)| = 1$ for all $1 \leq \ell \leq k$; write $D(\mathbf{p}^\ell) = \{\mathbf{x}^\ell\}$.

We next show that for each $1 \leq \ell \leq k$, the difference $\mathbf{x}^\ell - \mathbf{x}^{\ell-1}$ is a multiple of an element of $\mathcal{D}$. By construction, the line segment with endpoints $\mathbf{p}^{\ell-1}$ and $\mathbf{p}^\ell$ meets L at exactly one point—namely $\hat{\mathbf{p}}^\ell \in L \smallsetminus W$. Due to the upper hemicontinuity of demand, we must have that $\{\mathbf{x}^{\ell-1}, \mathbf{x}^\ell\} \subseteq D(\hat{\mathbf{p}}^\ell)$. Hence, the "only if" direction of Fact B.1 guarantees that $\mathbf{x}^\ell - \mathbf{x}^{\ell-1}$ is a multiple of an element of $\mathcal{D}$.

It remains to prove that $\mathbf{t} \cdot (\mathbf{x}^\ell - \mathbf{x}^{\ell-1}) < 0$ for all $1 \leq \ell \leq k$. We first prove that $\mathbf{x}^\ell \neq \mathbf{x}^{\ell-1}$. Suppose for sake of deriving a contradiction that $\mathbf{x}^\ell = \mathbf{x}^{\ell-1}$. In this case, since $\hat{\mathbf{p}}^\ell \in L$, there must exist $\mathbf{x} \in D(\hat{\mathbf{p}}^\ell) \smallsetminus \{\mathbf{x}^{\ell-1}, \mathbf{x}^\ell\}$. Lemma A.1(b) would then imply that $(\hat{\mathbf{p}}^\ell - \mathbf{p}^{\ell-1}) \cdot (\mathbf{x} - \mathbf{x}^{\ell-1}) < 0$ and that $(\hat{\mathbf{p}}^\ell - \mathbf{p}^\ell) \cdot (\mathbf{x} - \mathbf{x}^\ell) < 0$. Since $\mathbf{p}^{\ell-1}$, $\hat{\mathbf{p}}^\ell$, and $\mathbf{p}^\ell$ lie along the line segment with endpoints $\mathbf{p}$ and $\mathbf{p} + \mathbf{t}$ in that order, the price changes $\hat{\mathbf{p}}^\ell - \mathbf{p}^{\ell-1}$ and $\mathbf{p}^\ell - \hat{\mathbf{p}}^\ell$ must each be positive multiples of $\mathbf{t}$. Hence, we have that $\mathbf{t} \cdot (\mathbf{x} - \mathbf{x}^{\ell-1}) < 0$ and that $(-\mathbf{t}) \cdot (\mathbf{x} - \mathbf{x}^\ell) < 0$—contradicting the hypothesis that $\mathbf{x}^\ell = \mathbf{x}^{\ell-1}$. We can therefore conclude that $\mathbf{x}^\ell \neq \mathbf{x}^{\ell-1}$. Lemma 1′ then guarantees that $(\mathbf{p}^\ell - \mathbf{p}^{\ell-1}) \cdot (\mathbf{x}^\ell - \mathbf{x}^{\ell-1}) < 0$, and it follows that $\mathbf{t} \cdot (\mathbf{x}^\ell - \mathbf{x}^{\ell-1}) < 0$. □

To complete the proof of the "if" direction, let $\mathbf{p}' = (p'_i, \mathbf{p}_{I \smallsetminus \{i\}})$, let $\mathbf{t} = \mathbf{p}' - \mathbf{p}$, and let S be as in Lemma B.1. As demand is upper hemicontinuous, there exists a vector $\mathbf{s} \in \mathbb{R}^I$ such that $\mathbf{p} + \mathbf{s} \in S$, $D(\mathbf{p} + \mathbf{s}) = \{\mathbf{x}\}$, and $D(\mathbf{p}' + \mathbf{s}) = \{\mathbf{x}'\}$. Applying Lemma B.1 to $\mathbf{p}^0 = \mathbf{p} + \mathbf{s}$, we see that $\mathbf{x}' - \mathbf{x}$ must be a nonnegative linear combination of elements of $\mathcal{D}_i^-$.

Proof of the "if" direction. The argument uses the "if" direction of Fact B.1 to show that V is of demand type $\mathcal{D}$. As in the statement of Fact B.1, let $\hat{\mathbf{p}}$ be a price vector such that Conv $D(\hat{\mathbf{p}})$ is a line segment. Suppose that Conv $D(\hat{\mathbf{p}})$ has endpoints $\mathbf{x}$ and $\mathbf{x}'$; we wish to to show that $\mathbf{x}' - \mathbf{x}$ is parallel to an element of $\mathcal{D}$.

Consider a good i such that

$$i \in \arg\min_{j \mid x'_j \neq x_j} |x'_j - x_j|. \tag{B.2}$$

Exchanging the roles of $\mathbf{x}$ and $\mathbf{x}'$ if necessary, we can assume that $x'_i - x_i < 0$. By Claim B.1, there exists $\lambda > 0$ such that letting $\mathbf{p} = (\hat{p}_i - \lambda, \hat{\mathbf{p}}_{I \smallsetminus \{i\}})$ and $p'_i = \hat{p}_i + \lambda$, we have that $D(\mathbf{p}) = \{\mathbf{x}\}$ and that $D(p'_i, \mathbf{p}_{I \smallsetminus \{i\}}) = \{\mathbf{x}'\}$.

The hypothesis of the "if" direction of the theorem thus entails that there exist constants $\alpha_{\mathbf{d}} \geq 0$ for $\mathbf{d} \in \mathcal{D}_i^-$ such that

$$\mathbf{x}' - \mathbf{x} = \sum_{\mathbf{d} \in \mathcal{D}_i^-} \alpha_{\mathbf{d}} \mathbf{d}. \tag{B.3}$$

Let $S = \{\mathbf{d} \in \mathcal{D}_i^- \mid \alpha_{\mathbf{d}} > 0\}$. Since $\mathcal{D}$ is consistent, we have that $\mathcal{D} \subseteq \{-1, 0, 1\}^I$ and hence that $d_i = -1$ for all $\mathbf{d} \in \mathcal{D}_i^-$. Since $\mathcal{D}$ is closed under negation, consistency also entails that for each good $j \neq i$, we either have that $d_j \in \{0, 1\}$ for all $\mathbf{d} \in \mathcal{D}_i^-$ or that $d_j \in \{-1, 0\}$ for all $\mathbf{d} \in \mathcal{D}_i^-$. In either case, we have that

$$|x'_j - x_j| = \left| \sum_{\mathbf{d} \in \mathcal{D}_i^-} \alpha_{\mathbf{d}} d_j \right| = \sum_{\mathbf{d} \in \mathcal{D}_i^-} \alpha_{\mathbf{d}} |d_j| \leq \sum_{\mathbf{d} \in \mathcal{D}_i^-} \alpha_{\mathbf{d}} = |x'_i - x_i|,$$

with equality if and only if either $d_j = 1$ for all $\mathbf{d} \in S$, or $d_j = -1$ for all $\mathbf{d} \in S$. If the inequality holds strictly, then (B.2) implies that $x'_j = x_j$—in which case $d_j = 0$ must hold for all $\mathbf{d} \in S$.

Thus, for each good $j \neq i$, we either have that $d_j = -1$ for all $\mathbf{d} \in S$, that $d_j = 1$ for all $\mathbf{d} \in S$, or that $d_j = 0$ for all $\mathbf{d} \in S$. Hence, we must have that $|S| = 1$, in which case (B.3) implies that $\mathbf{x}' - \mathbf{x}$ is proportional to an element of $\mathcal{D}$. Since $\hat{\mathbf{p}}$ was arbitrary, the "if" direction of the theorem therefore follows from the "if" direction of Fact B.1.

B.5 Proof of Proposition 1

We first show that $\mathcal{D} \subseteq \{-1, 0, 1\}^I$. Let $\mathbf{d} \in \mathcal{D}$ be arbitrary, and suppose that $d_i \neq 0$. Let $S = \{\mathbf{d}\} \cup \{\mathbf{e}^j \mid j \neq i\}$, which is a basis of $\mathbb{R}^I$. Unimodularity requires that the matrix whose columns are the elements of S is ± 1; this determinant is clearly $\pm d_i$, and we thus have that $d_i = \pm 1$. Since $\mathbf{d}$ and i were arbitrary, we must have that $\mathcal{D} \subseteq \{-1, 0, 1\}^I$.

Now let $\mathbf{d}, \mathbf{d}' \in \mathcal{D}$ and let $i \neq j$ be goods. Suppose that $d_i d_j > 0$ and that $d'_i d'_j \leq 0$. We need to show that $d'_i d'_j = 0$. Since $\mathcal{D}$ is closed under negation, we can assume without loss of generality that $d_i = d_j = d'_i = 1$ and that $d'_j \leq 0$. Consider the set

$$S = \{\mathbf{d}, \mathbf{d}'\} \cup \{\mathbf{e}^k \mid k \in I \smallsetminus \{i, j\}\},$$

which is a basis of $\mathbb{R}^I$. Unimodularity requires that the matrix whose columns are the elements of S have determinant ± 1; this determinant is clearly $\pm(1 - d'_j)$, and hence we must have that $d'_j = 0$. In particular, we have that $d'_i d'_j = 0$—as desired.

B.6 Proof of Lemma 1′

Lemma 1′ is the special case of Lemma A.1(b) in which $D(\mathbf{p}') = \{\mathbf{x}'\}$.

B.7 Proof of Theorem 3

Proof of the "only if" direction. Let $\mathbf{t} = \mathbf{p}' - \mathbf{p}$, and let S be as in Lemma B.1. As demand is upper hemicontinuous, there exists a vector $\mathbf{s} \in \mathbb{R}^I$ such that $\mathbf{p} + \mathbf{s} \in S$, $D(\mathbf{p} + \mathbf{s}) = \{\mathbf{x}\}$, and $D(\mathbf{p}' + \mathbf{s}) = \{\mathbf{x}'\}$. Applying Lemma B.1 to $\mathbf{p}^0 = \mathbf{p} + \mathbf{s}$, we see that $\mathbf{x}' - \mathbf{x}$ must be a nonnegative linear combination of elements of $\mathcal{D}(\mathbf{p} + \mathbf{s}, \mathbf{p}' + \mathbf{s}) = \mathcal{D}(\mathbf{p}, \mathbf{p}')$.

Proof of the "if" direction. Suppose for sake of deriving a contradiction that V is not of demand type $\mathcal{D}$. By (the contrapositive of) the "if" direction of Fact B.1, there must exist a price vector $\hat{\mathbf{p}}$ such that $\operatorname{Conv} D(\hat{\mathbf{p}})$ is a line segment with endpoints $\mathbf{x}$ and $\mathbf{x}'$, where $\mathbf{x}' - \mathbf{x}$ is not parallel to an element of $\mathcal{D}$. Since $\mathcal{D} \subseteq \mathbb{Z}^I$, there must exist a vector $\mathbf{s} \in \mathbb{R}^I$ such that $\mathbf{s} \cdot (\mathbf{x}' - \mathbf{x}) = 0$ but $\mathbf{s} \cdot \mathbf{d} \neq 0$ for all $\mathbf{d} \in \mathcal{D}$. Since $\mathcal{D}$ is finite, there exists $\epsilon > 0$ such that $|\mathbf{s} \cdot \mathbf{d}| > \epsilon |(\mathbf{x}' - \mathbf{x}) \cdot \mathbf{d}|$ for all $\mathbf{d} \in \mathcal{D}$. Letting $\mathbf{t} = \mathbf{s} - \epsilon(\mathbf{x}' - \mathbf{x})$, we have that $\mathbf{t} \cdot (\mathbf{x}' - \mathbf{x}) < 0$; and for all $\mathbf{d} \in \mathcal{D}$, that $\mathbf{t} \cdot \mathbf{d} \neq 0$ and that the quantities $\mathbf{s} \cdot \mathbf{d}$ and $\mathbf{t} \cdot \mathbf{d}$ have the same sign.

By Claim B.1, there exists $\lambda > 0$ such that $D(\hat{\mathbf{p}} - \lambda \mathbf{t}) = \{\mathbf{x}\}$ and $D(\hat{\mathbf{p}} + \lambda \mathbf{t}) = \{\mathbf{x}'\}$. Let $\mathbf{p} = \hat{\mathbf{p}} - \lambda \mathbf{t}$ and let $\mathbf{p}' = \hat{\mathbf{p}} + \lambda \mathbf{t}$. By hypothesis, there exist constants $\alpha_{\mathbf{d}} \geq 0$ for

$\mathbf{d} \in \mathcal{D}(\mathbf{p}, \mathbf{p}')$ such that

$$\mathbf{x}' - \mathbf{x} = \sum_{\mathbf{d} \in \mathcal{D}(\mathbf{p}, \mathbf{p}')} \alpha_{\mathbf{d}} \mathbf{d}.$$

In particular, we have that

$$0 = \mathbf{s} \cdot (\mathbf{x}' - \mathbf{x}) = \sum_{\mathbf{d} \in \mathcal{D}(\mathbf{p}, \mathbf{p}')} \alpha_{\mathbf{d}} \mathbf{s} \cdot \mathbf{d}.$$

By the construction of $\mathbf{t}$, we have that $\mathbf{s} \cdot \mathbf{d} < 0$ for all $\mathbf{d} \in \mathcal{D}(\mathbf{p}, \mathbf{p}')$. Hence, we must have that $\alpha_{\mathbf{d}} = 0$ for all $\mathbf{d} \in \mathcal{D}(\mathbf{p}, \mathbf{p}')$, which contradicts the fact that $\mathbf{x}' \neq \mathbf{x}$. Therefore, we can conclude that V must be of demand type $\mathcal{D}$.

B.8 Proof of Corollary 3

The "if" direction follows from the corresponding direction of Theorem 3. For the "only if" direction, let M be such that $X \subseteq \left[-\frac{M}{2}, \frac{M}{2}\right]$. Fact B.1 implies that for all price vectors $\mathbf{p}$ for which $\operatorname{Conv} D(\mathbf{p})$ is a line segment, $\operatorname{Conv} D(\mathbf{p})$ is parallel to an element of $\mathcal{D}$. As demand at each price vector is a subset of X, Fact B.1 implies that for all price vectors $\mathbf{p}$ for which $\operatorname{Conv} D(\mathbf{p})$ is a line segment, $\operatorname{Conv} D(\mathbf{p})$ is in fact parallel to an element of $\mathcal{D} \cap [-M, M]$. Hence, Fact B.1 implies that V is of demand type $\mathcal{D} \cap [-M, M]$. The "only if" direction of the corollary thus follows from the corresponding direction of Theorem 3.

B.9 Proof of Proposition 2

The proof uses the following technical claim.

Claim B.2. *Let* $\mathbf{p}$ *be a price vector and let* $\epsilon > 0$. *For each extreme point* $\mathbf{x}$ *of* $\operatorname{Conv} D(\mathbf{p})$, *there exists a vector* $\mathbf{s} \in \mathbb{R}^I$ *such that* $\|\mathbf{s}\| \leq \epsilon$ *and* $D(\mathbf{p} - \mathbf{s}) = \{\mathbf{x}\}$.

Proof. As demand is upper hemi-continuous, by reducing ϵ if necessary, we can assume that $D(\hat{\mathbf{p}}) \subseteq D(\mathbf{p})$ for all price vectors $\hat{\mathbf{p}}$ with $\|\hat{\mathbf{p}} - \mathbf{p}\| \leq \epsilon$.

By construction, we must have that $\mathbf{x} \in D(\mathbf{p})$. Since $\operatorname{Conv} D(\mathbf{p})$ is a polytope, $\mathbf{x}$ must be a vertex of $\operatorname{Conv} D(\mathbf{p})$—i.e., there must exist a vector $\mathbf{s} \in \mathbb{R}^I$ such that

$$\{\mathbf{x}\} = \underset{\mathbf{x}' \in \operatorname{Conv} D(\mathbf{p})}{\arg\max} \{\mathbf{s} \cdot \mathbf{x}'\}.$$

In particular, we have that $\mathbf{s} \cdot (\mathbf{x}' - \mathbf{x}) > 0$ for all $\mathbf{x}' \in \operatorname{Conv} D(\mathbf{p}) \smallsetminus \{\mathbf{x}\}$.

Without loss of generality, we can assume that $\|\mathbf{s}\| = \epsilon$. Applying Lemma A.1(a) to the price change from $\mathbf{p} - \mathbf{s}$ to $\mathbf{p}$, we see that $\mathbf{s} \cdot (\mathbf{x}' - \mathbf{x}) \leq 0$ for all $\mathbf{x}' \in D(\mathbf{p} - \mathbf{s})$. It follows that $\mathbf{x}' \notin D(\mathbf{p} - \mathbf{s})$ for all $\mathbf{x}' \in D(\mathbf{p}) \smallsetminus \{\mathbf{x}\}$. Since $\|\mathbf{s}\| = \epsilon$, we have that $D(\mathbf{p} - \mathbf{s}) \subseteq D(\mathbf{p})$, and it follows that $D(\mathbf{p} - \mathbf{s}) = \{\mathbf{x}\}$—as desired. $\square$

By the Krein-Millman Theorem, we can write

$$\mathbf{x} = \sum_{\ell=1}^{m} \alpha_\ell \mathbf{x}^\ell,$$

where $\mathbf{x}^1, \ldots, \mathbf{x}^m$ are extreme points of $\operatorname{Conv} D(\mathbf{p})$ and $\alpha_1, \ldots, \alpha_m$ are nonnegative real numbers with $\sum_{\ell=1}^{m} \alpha_\ell = 1$.

Let ϵ be such that $D(\mathbf{p} - \mathbf{s}) \subseteq D(\mathbf{p})$ and $D(\mathbf{p}' - \mathbf{s}) \subseteq D(\mathbf{p}')$ for all vectors $\mathbf{s} \in \mathbb{R}^I$ with $\|\mathbf{s}\| \leq \epsilon$; such an ϵ exists due to the upper hemicontinuity of demand. For each $1 \leq \ell \leq m$, let $\mathbf{s}^\ell \in \mathbb{R}^I$ be such that $\|\mathbf{s}^\ell\| \leq \epsilon$ and $D(\mathbf{p} - \mathbf{s}^\ell) = \{\mathbf{x}^\ell\}$; such vectors exist by Claim B.2. Let ϵ^ℓ be such that $D(\mathbf{p} - \mathbf{s}^\ell - \mathbf{t}^\ell) = \{\mathbf{x}^\ell\}$ for all vectors $\mathbf{t}^\ell \in \mathbb{R}^I$ with $\|\mathbf{t}^\ell\| \leq \epsilon^\ell$; such an ϵ^ℓ exists due to the upper hemicontinuity of demand. Letting $\mathbf{x}'^\ell$ be an arbitrary extreme point of $\operatorname{Conv} D(\mathbf{p}' - \mathbf{s}^\ell)$, Claim B.2 (applied to the price vector $\mathbf{p}' - \mathbf{s}^\ell$) guarantees that there exists a vector $\mathbf{t}^\ell \in \mathbb{R}^I$ with $\|\mathbf{t}^\ell\| \leq \epsilon^\ell$ such that $D(\mathbf{p}' - \mathbf{s}^\ell - \mathbf{t}^\ell) = \{\mathbf{x}'^\ell\}$.

By construction, we have that $D(\mathbf{p} - \mathbf{s}^\ell - \mathbf{t}^\ell) = \{\mathbf{x}^\ell\}$, and the "only if" direction of Corollary 3 (applied to the price vectors $\mathbf{p} - \mathbf{s}^\ell - \mathbf{t}^\ell$ and $\mathbf{p}' - \mathbf{s}^\ell - \mathbf{t}^\ell$) therefore guarantees that $\mathbf{x}'^\ell - \mathbf{x}^\ell$ is a nonnegative linear combination of elements of $\mathcal{D}(\mathbf{p} - \mathbf{s}^\ell - \mathbf{t}^\ell, \mathbf{p}' - \mathbf{s}^\ell - \mathbf{t}^\ell) = \mathcal{D}(\mathbf{p}, \mathbf{p}')$. By the definition of ϵ, we have that $D(\mathbf{p}' - \mathbf{s}^\ell) \subseteq D(\mathbf{p}')$, and hence in particular that $\mathbf{x}'^\ell \in \operatorname{Conv} D(\mathbf{p}')$, for $1 \leq \ell \leq m$. Letting

$$\mathbf{x}' = \sum_{\ell=1}^{m} \alpha_\ell \mathbf{x}'^\ell \in \operatorname{Conv} D(\mathbf{p}'),$$

we have that

$$\mathbf{x}' - \mathbf{x} = \sum_{\ell=1}^{m} \alpha_\ell \left(\mathbf{x}'^\ell - \mathbf{x}^\ell \right),$$

so $\mathbf{x}' - \mathbf{x}$ must be a nonnegative linear combination of elements of $\mathcal{D}(\mathbf{p}, \mathbf{p}')$—as desired.

B.10 Proof of Corollary 4

The "if" direction follows from the corresponding direction of Theorem 3, as every unimodular $\mathcal{D}$ is finite (see, e.g., Korkine and Zolotareff (1877)).[23]

It remains to prove the "only if" direction. Let $\mathbf{p}, \mathbf{p}'$ be price vectors, and let $\mathbf{x} \in D(\mathbf{p})$. By Proposition 2, there exists $\mathbf{x}'' \in \operatorname{Conv} D(\mathbf{p}')$ such that $\mathbf{x}'' - \mathbf{x}$ is a nonnegative linear combination of elements of $\mathcal{D}(\mathbf{p}, \mathbf{p}')$. Write

$$\mathbf{x}'' - \mathbf{x} = \sum_{\mathbf{d} \in \mathcal{D}(\mathbf{p}, \mathbf{p}')} \alpha_{\mathbf{d}} \mathbf{d},$$

where $\alpha_{\mathbf{d}} \geq 0$ for all $\mathbf{d} \in \mathcal{D}(\mathbf{p}, \mathbf{p}')$. Let M be an integer with $M \geq \alpha_{\mathbf{d}}$ for all $\mathbf{d} \in \mathcal{D}(\mathbf{p}, \mathbf{p}')$.

For each $\mathbf{d} \in \mathcal{D}(\mathbf{p}, \mathbf{p}')$, consider the set

$$\widehat{X}_{\mathbf{d}} = \{\mathbf{0}, -\mathbf{d}, -2\mathbf{d}, \ldots, -M\mathbf{d}\}$$

and the valuation $\widehat{V}_{\mathbf{d}} : \widehat{X}_{\mathbf{d}} \to \mathbb{R}$ defined by $\widehat{V}_{\mathbf{d}}(\mathbf{y}) = \mathbf{p}' \cdot \mathbf{y}$. Let $\widehat{D}_{\mathbf{d}}$ be the demand correspondence for the valuation $\widehat{V}_{\mathbf{d}}$; by construction, we have that

$$\widehat{D}_{\mathbf{d}}(\mathbf{p}') = \widehat{X}_{\mathbf{d}} = \operatorname{Conv} \widehat{X}_{\mathbf{d}} \cap \mathbb{Z}^I. \tag{B.4}$$

Hence, each valuation $\widehat{V}_{\mathbf{d}}$ is concave. Valuation $\widehat{V}_{\mathbf{d}}$ is of demand type $\mathcal{D}$ by Theorem 3 since a nonzero change in demand between price vectors $\mathbf{p}, \mathbf{p}'$ (at which demand is unique) is a multiple of $\mathbf{d}$ (by construction) with negative inner product with $\mathbf{p}' - \mathbf{p}$ (by Lemma 1′)—hence a nonnegative linear combination of elements of $\mathcal{D}(\mathbf{p}, \mathbf{p}')$.

It also follows from (B.4) that

$$\mathbf{x} = \mathbf{x}'' + \sum_{\mathbf{d} \in \mathcal{D}(\mathbf{p}, \mathbf{p}')} (-\alpha_{\mathbf{d}} \mathbf{d}) \in \operatorname{Conv} D(\mathbf{p}') + \sum_{\mathbf{d} \in \mathcal{D}(\mathbf{p}, \mathbf{p}')} \operatorname{Conv} \widehat{D}_{\mathbf{d}}(\mathbf{p}').$$

Thus, $\mathbf{p}'$ is a pseudoequilibrium price vector (in the sense of Milgrom and Strulovici (2009)) in the economy with agents of valuations V and $\widehat{V}_{\mathbf{d}}$ for $\mathbf{d} \in \mathcal{D}(\mathbf{p}, \mathbf{p}')$ when the total supply of goods is $\mathbf{x}$. Corollary 4.4 in BK19 implies that a competitive equilibrium exists in that economy. Hence, by Theorem 18 in Milgrom and Strulovici (2009), $\mathbf{p}$ must be a competitive equilibrium price vector in that economy—i.e., we

[23] In fact, Korkine and Zolotareff (1877) showed that if $\mathcal{D}$ is unimodular, then $|\mathcal{D}| \leq |I|^2 + |I|$.

must have that

$$\mathbf{x} \in D(\mathbf{p}') + \sum_{\mathbf{d} \in \mathcal{D}(\mathbf{p},\mathbf{p}')} \widehat{D}_{\mathbf{d}}(\mathbf{p}').$$

Writing

$$\mathbf{x} = \mathbf{x}' + \sum_{\mathbf{d} \in \mathcal{D}(\mathbf{p},\mathbf{p}')} (-\beta_{\mathbf{d}} \mathbf{d}),$$

where $\mathbf{x}' \in D(\mathbf{p}')$ and $\beta_{\mathbf{d}} \in \{0, 1, \ldots, M\}$ for $\mathbf{d} \in \mathcal{D}(\mathbf{p}, \mathbf{p}')$, we have that

$$\mathbf{x}' - \mathbf{x} = \sum_{\mathbf{d} \in \mathcal{D}(\mathbf{p},\mathbf{p}')} \beta_{\mathbf{d}} \mathbf{d},$$

is a nonnegative linear combination of elements of $\mathcal{D}(\mathbf{p}, \mathbf{p}')$—as desired.

C Additional Examples

This appendix presents additional examples illustrating the role of the hypotheses of Theorems 1–3 and Proposition 2 in the conclusions of those results.

The first example shows that the conclusion of the "only if" direction of Theorem 1 does not extend to simultaneous changes in the prices of several goods.

Example C.1 (Failure of Theorem 1 if multiple goods' prices can change simultaneously). There are two goods ($I = \{1, 2\}$). Let $X = \{0, 1\}^2$, and define $V : X \to \mathbb{R}$ by $V(\mathbf{x}) = x_1 + x_2$.

Let $\mathcal{D} = \pm\{(1, 0), (0, 1)\}$. As discussed in the introduction, changing the price of either good (holding fixed the price of the other good) between prices at which demand is unique would lead to a change in demand of $\mathbf{0}$, $\pm(1, 0)$, or $\pm(0, 1)$. Hence, by Theorem 1, valuation V is of demand type $\mathcal{D}$.

However, letting $\mathbf{p} = (0, 0)$ and $\mathbf{p}' = (2, 2)$, we have that $D(\mathbf{p}) = \{(1, 1)\}$ and that $D(\mathbf{p}') = \{(0, 0)\}$. The difference $(1, 1) - (0, 0) = (1, 1)$ is not an element of $\mathcal{D}$.

The second example shows that the conclusion of Theorem 2 can fail if the second part of Definition 2 is not satisfied—i.e., if there are vectors $\mathbf{d}, \mathbf{d}' \in \mathcal{D}$ and goods i, j with $d_i d_j > 0$ but $d'_i d'_j < 0$. (Example 1 showed a similar result if the first part of Definition 2 is not satisfied—i.e., if $\mathcal{D} \not\subseteq \{-1, 0, 1\}^I$.)

Example C.2 (Failure of Theorem 2 for an inconsistent $\mathcal{D} \subseteq \{-1, 0, 1\}^I$). There are two goods ($I = \{1, 2\}$). Let $X = \{(0, 0), (1, 0), (2, 0)\}$, and define $V : X \to \mathbb{R}$ by

$V(\mathbf{x}) = x_1$.

Let $\mathcal{D} = \pm\{(1,1),(1,-1)\}$. Note that $\mathcal{D}$ is not consistent as, letting $\mathbf{d} = (1,1) \in \mathcal{D}$ and $\mathbf{d}' = (1,-1) \in \mathcal{D}$, we have that $d_1 d_2 > 0$ but $d'_1 d'_2 < 0$. Valuation V is not of demand type $\mathcal{D}$. Indeed, the set of price vectors at which demand is nonunique is

$$L = \{\mathbf{p} \mid p_1 = 1\}.$$

Since bundles $(0,0)$ and $(2,0)$ are both demanded at each price vector in L, the set L is a LIP facet. And the normal to L is in the direction of $(1,0)$—which is not an element of $\mathcal{D}$.

However, the hypothesis of the "if" direction of Theorem 2 does hold. Indeed, let $\mathbf{p}$ be a price vector, let $p'_i > p_i$ be a new price for a good i, and suppose that $D(\mathbf{p}) = \{\mathbf{x}\}$ and that $D(p'_i, \mathbf{p}_{I \smallsetminus \{i\}}) = \{\mathbf{x}'\}$. By the law of demand, we must either have that $\mathbf{x} = \mathbf{x}'$, in which case the difference $\mathbf{x}' - \mathbf{x}$ is trivially a nonnegative linear combination of elements of $\mathcal{D}_i^-$; or that $\mathbf{x} = (2,0)$, $\mathbf{x}' = (0,0)$, and that $i = 1$. As $\mathcal{D}_1^- = \{(-1,-1),(-1,1)\}$ and

$$(-2,0) = (-1,-1) + (-1,1),$$

the difference $\mathbf{x}' - \mathbf{x}$ is a nonnegative linear combination of elements of $\mathcal{D}_i^-$ in the latter case as well.[24]

The third example shows that the conclusion of Theorem 3 can fail if $\mathcal{D}$ is infinite.

Example C.3 (Failure of Theorem 3 for infinite $\mathcal{D}$). Let I, X, and V be as in Example C.2.

Let

$$\mathcal{D} = \{(n, \pm 1) \mid n \in \mathbb{Z}\}.$$

Valuation V is not of demand type $\mathcal{D}$. Indeed, as shown in Example C.2, there is a LIP facet whose normal vectors are in the direction of $(1,0) \notin \mathcal{D}$.

However, the hypothesis of the "if" direction of Theorem 3 does hold. Indeed, let $\mathbf{p}, \mathbf{p}'$ be price vectors such that $D(\mathbf{p}) = \{\mathbf{x}\}$ and $D(\mathbf{p}') = \{\mathbf{x}'\}$. If $\mathbf{x}' = \mathbf{x}$, then

[24] In fact, in this example, the difference $\mathbf{x}' - \mathbf{x}$ is a nonnegative integer combination of elements of $\mathcal{D}_i^-$ in either case. A similar remark applies to Example 6. Hence, the conclusion of the "only if" direction of Theorem 2 would not generally hold if either part of Definition 2 were relaxed even under strengthening the hypothesis that $\mathbf{x}' - \mathbf{x}$ be a nonnegative linear combination of elements of $\mathcal{D}_i^-$ to require that $\mathbf{x}' - \mathbf{x}$ be a nonnegative integer combination of elements of $\mathcal{D}_i^-$.

$\mathbf{x}' - \mathbf{x} = \mathbf{0}$ is a nonnegative linear combination of elements of $\mathcal{D}(\mathbf{p}, \mathbf{p}')$. Hence, we can assume that $\mathbf{x}' \neq \mathbf{x}$. Without loss of generality, we can assume that $\mathbf{x} = (2, 0)$ and that $\mathbf{x}' = (0, 0)$. In this case, by the law of demand, we must have that $p_1 < p_1'$. Consider the ratio

$$r = \frac{p_2' - p_2}{p_1' - p_1},$$

and let n be an integer such that $n \leq r < n+1$. Consider the vectors $\mathbf{d}^1 = (n-1, -1)$ and $\mathbf{d}^2 = (-n-1, 1)$, which are both elements of $\mathcal{D}$. Note that

$$(\mathbf{p}' - \mathbf{p}) \cdot \mathbf{d}^1 = (n-1)(p_1' - p_1) - (p_2' - p_2) = (n - r - 1)(p_1' - p_1) < 0$$
$$(\mathbf{p}' - \mathbf{p}) \cdot \mathbf{d}^2 = (-n-1)(p_1' - p_1) + (p_2' - p_2) = (r - n - 1)(p_1' - p_1) < 0.$$

Thus, we have that $\mathbf{d}^1, \mathbf{d}^2 \in \mathcal{D}(\mathbf{p}, \mathbf{p}')$. We also have that

$$\mathbf{x}' - \mathbf{x} = (-2, 0) = \mathbf{d}^1 + \mathbf{d}^2,$$

so $\mathbf{x}' - \mathbf{x}$ is a nonnegative linear combination of elements of $\mathcal{D}(\mathbf{p}, \mathbf{p}')$.[25]

The fourth example shows that the conclusion of Proposition 2 would not hold if the bundle $\mathbf{x}'$ were required to be integer or to be in $D(\mathbf{p}')$.

Example C.4 (Failure of Proposition 2 if $\mathbf{x}'$ were required to be integer or to be in $D(\mathbf{p}')$). There are two goods ($I = \{1, 2\}$). Let

$$X = \{(0,0,0), (0,1,1), (1,0,1), (1,1,0), (1,1,1), (1,1,2), (1,2,1)(2,1,1), (2,2,2)\},$$

and define $V : X \to \mathbb{R}$ by $V(\mathbf{x}) = x_1 + x_2 + x_3$.

Let $\mathcal{D} = \pm\{(0,1,1), (1,0,1), (1,1,0)\}$.[26] To show that V is of demand type $\mathcal{D}$, let $\mathcal{D}^+ = \{(0,1,1), (1,0,1), (1,1,0)\}$, and for each $\mathbf{d} \in \mathcal{D}^+$, define a set $\widehat{X}_{\mathbf{d}} = \{0, \mathbf{d}\}$ and a valuation $\widehat{V}_{\mathbf{d}} : \widehat{X}_{\mathbf{d}} \to \mathbb{R}$ by $\widehat{V}_{\mathbf{d}}(\mathbf{x}) = x_1 + x_2 + x_3$. It follows from Theorem 1 that each valuation $\widehat{V}_{\mathbf{d}}$ is of demand type $\mathcal{D}$. Letting $\widehat{D}_{\mathbf{d}}$ denote the demand correspondence for the valuation $\widehat{V}_{\mathbf{d}}$, we have that we have that $D(\mathbf{p}) = \sum_{\mathbf{d} \in \mathcal{D}^+} \widehat{D}_{\mathbf{d}}(\mathbf{p})$ for all price

[25] In fact, in this example, the difference $\mathbf{x}' - \mathbf{x}$ is a nonnegative integer combination of elements of $\mathcal{D}(\mathbf{p}, \mathbf{p}')$. Hence, the conclusion of the "only if" direction of Theorem 3 would not generally hold for infinite $\mathcal{D}$ even under strengthening the hypothesis that $\mathbf{x}' - \mathbf{x}$ be a nonnegative linear combination of elements of $\mathcal{D}_i^-$ to require that $\mathbf{x}' - \mathbf{x}$ be a nonnegative integer combination of elements of $\mathcal{D}_i^-$.

[26] This $\mathcal{D}$ is consistent. Hence, the example shows that Proposition 2 would not hold even for consistent $\mathcal{D}$ if $\mathbf{x}'$ were required to be in $D(\mathbf{p}')$.

vectors $\mathbf{p}$ with $|D(\mathbf{p})| = 1$ by construction. Hence, applying the "if" direction of Theorem 2 to each valuation $\widehat{V}_{\mathbf{d}}$, we see that for all goods i, price vectors $\mathbf{p}$, and new prices $p'_i > p_i$ such that $D(\mathbf{p}) = \{\mathbf{x}\}$ and $D(p'_i, \mathbf{p}_{I \smallsetminus \{i\}}) = \{\mathbf{x}'\}$, the difference $\mathbf{x}' - \mathbf{x}$ is a nonnegative linear combination of elements of $\mathcal{D}_i^-$. Thus, by the "only if" direction of Theorem 2, V is of demand type $\mathcal{D}$.

By construction, demand at the price vector $\mathbf{p} = (1,1,1)$ is $D(\mathbf{p}) = X$. In particular, we have that $\mathbf{x} = (1,1,1) \in D(\mathbf{p})$.

Consider the new price $p'_1 = 2$ for good 1. By construction, we have that $D(p'_1, p_2, p_3) = \{(0,0,0), (0,1,1)\}$. But as $\mathcal{D}_1^- = \{(-1,-1,0), (-1,0,-1)\}$, the difference $\mathbf{x}' - \mathbf{x}$ is not a nonnegative linear combination of elements of $\mathcal{D}_1^-$ for any $\mathbf{x}' \in D(p'_1, p_2, p_3)$—or even for any integer bundle $\mathbf{x}' \in \operatorname{Conv} D(p'_1, p_2, p_3)$. Indeed, neither the difference $(0,0,0) - (1,1,1) = (-1,-1,-1)$ nor the difference $(0,1,1) - (1,1,1) = (-1,0,0)$ is a linear combination of elements of $\mathcal{D}_1^-$.[27]

Furthermore, even if it is possible to take $\mathbf{x}' \in D(\mathbf{p}')$ in Proposition 2, it may not be possible to express $\mathbf{x}' - \mathbf{x}$ as a nonnegative integer combination of elements of $\mathcal{D}(\mathbf{p}, \mathbf{p}')$—unlike in Theorems 2 and 3 and Corollary 3 (see Section 4.2 and Footnote 20).

Example C.5 (Possibility that $\mathbf{x}' - \mathbf{x}$ cannot be expressed as an integer combination of elements of $\mathcal{D}(\mathbf{p}, \mathbf{p}')$ in Proposition 2 even if $\mathbf{x}'$ can be taken to be an element of $D(\mathbf{p}')$). Let I, X, V, $\mathbf{p}$, and $\mathbf{x}$ be as in Example C.4. Consider on the new price vector $\mathbf{p}' = (2,2,2)$. By the law of demand, we have that $D(\mathbf{p}') = \{(0,0,0)\}$; let $\mathbf{x}' = (0,0,0)$. As

$$\mathcal{D}(\mathbf{p}, \mathbf{p}') = \{(-1,-1,0), (-1,0,-1), (0,-1,-1)\}$$

[27] On the other hand, as guaranteed by Proposition 2, there exists $\mathbf{x}' \in \operatorname{Conv} D(p'_1, p_2, p_3)$ such that $\mathbf{x}' - \mathbf{x}$ is a nonnegative linear combination of elements of $\mathcal{D}_1^-$. Indeed, letting $\mathbf{x}' = \left(0, \frac{1}{2}, \frac{1}{2}\right) \in \operatorname{Conv} D(p'_1, p_2, p_3)$, we have that

$$\mathbf{x}' - \mathbf{x} = \left(-1, -\frac{1}{2}, -\frac{1}{2}\right) = \frac{1}{2}(-1,-1,0) + \frac{1}{2}(-1,0,1).$$

the unique expression of $\mathbf{x}' - \mathbf{x}$ as a linear combination of elements of $\mathcal{D}(\mathbf{p}, \mathbf{p}')$ is[28]

$$\mathbf{x}' - \mathbf{x} = (-1,-1,-1) = \frac{1}{2}(-1,-1,0) + \frac{1}{2}(-1,0,-1) + \frac{1}{2}(0,-1,-1).$$

So while the difference $\mathbf{x}' - \mathbf{x}$ is a nonnegative linear combination of elements of $\mathcal{D}(\mathbf{p}, \mathbf{p}')$, it is not a nonnegative integer combination of elements of $\mathcal{D}(\mathbf{p}, \mathbf{p}')$.

D Characterizing Consistent $\mathcal{D}$

In this appendix, we formally state and prove the characterization of consistent $\mathcal{D}$ described in Footnotes 12 and 14 in Section 4.2.

We first show that $\mathcal{D} \subseteq \{-1, 0, 1\}^I$ holds if and only if for all valuations of demand type $\mathcal{D}$, increasing the price of a good i always makes demand for other goods change, in magnitude, by the amount by which demand for i falls—as asserted in Footnote 12.

Proposition D.1. *We have that $\mathcal{D} \subseteq \{-1, 0, 1\}^I$ if and only if for all valuations V of demand type $\mathcal{D}$, goods i and j, price vectors $\mathbf{p}$, and new prices $p'_i > p_i$ such that $D(\mathbf{p}) = \{\mathbf{x}\}$ and $D(p'_i, \mathbf{p}_{I \smallsetminus \{i\}}) = \{\mathbf{x}'\}$, we have that $|x'_j - x_j| \le |x'_i - x_i|$.*

Proof. The "only if" direction follows from the corresponding direction of Corollary 3. To prove the "if" direction, we prove the contrapositive. Suppose that $\mathcal{D} \not\subseteq \{-1, 0, 1\}^I$; we prove that there is a valuation V of demand type $\mathcal{D}$, goods i and j, a price vector $\mathbf{p}$, and a new price $p'_i > p_i$ such that $D(\mathbf{p}) = \{\mathbf{x}\}$ and $D(p'_i, \mathbf{p}_{I \smallsetminus \{i\}}) = \{\mathbf{x}'\}$ but $|x'_j - x_j| > |x'_i - x_i|$.

Let $\mathbf{d} \in \mathcal{D} \smallsetminus \{-1, 0, 1\}^I$. Since $\mathbf{d}$ is primitive, there must exist goods i and j such that $|d_j| > |d_i| > 0$. By negating $\mathbf{d}$ if necessary, we can assume that $d_i < 0$.

Let $X = \{\mathbf{0}, \mathbf{d}\}$ and define a valuation $V : X \to \mathbb{R}$ by $V(\mathbf{x}) = 0$. Valuation V is of demand type $\mathcal{D}$ by Corollary 3 since a nonzero change in demand between price vectors $\mathbf{p}, \mathbf{p}'$ (at which demand is unique) is a multiple of $\mathbf{d}$ (by construction) with negative inner product with $\mathbf{p}' - \mathbf{p}$ (by Lemma 1′).

Note that $D(\mathbf{0}) = \{\mathbf{0}, \mathbf{d}\}$ by construction. By Claim B.2, there exists $\lambda > 0$ such that letting $\mathbf{p} = ((-\lambda)_i, \mathbf{0}_{I \smallsetminus \{i\}})$ and $p'_i = \lambda$, we have that $D(\mathbf{p}) = \{\mathbf{0}\}$ and $D(p'_i, \mathbf{p}_{I \smallsetminus \{i\}}) = \{\mathbf{d}\}$. To complete the proof, we simply note that $|d_j| > |d_i|$ holds by the definitions of the goods i, j. □

[28]The uniqueness of this expression is immediate as the vectors in $\mathcal{D}(\mathbf{p}, \mathbf{p}')$ are linearly independent.

We next show that given goods i, j, the product $d_i d_j$ is nonnegative (resp. nonpositive) for all $\mathbf{d} \in \mathcal{D}$ if and only if increasing the price of i always weakly lowers (resp. weakly raises) demand for all valuations of demand type $\mathcal{D}$—as asserted in Footnote 14.

Proposition D.2. *Let i and j be goods. The product $d_i d_j$ is nonnegative (resp. nonpositive) for all $\mathbf{d} \in \mathcal{D}$ or nonpositive for all $\mathbf{d} \in D$ if and only if for all valuations V of demand type $\mathcal{D}$, price vectors $\mathbf{p}$ and new prices $p'_i > p_i$ such that $D(\mathbf{p}) = \{\mathbf{x}\}$ and $D(p'_i, \mathbf{p}_{I \smallsetminus \{i\}}) = \{\mathbf{x}'\}$, we have that $x'_j \leq x_j$ (resp. $x'_j \geq x_j$).*

Proof. The "only if" direction follows from the corresponding direction of Corollary 3. To prove the "if" direction, we prove the contrapositive. Suppose that there exists $\mathbf{d} \in \mathcal{D}$ such that $d_i d_j$ is negative (resp. positive); we prove that there is a valuation V of demand type $\mathcal{D}$, a price vector $\mathbf{p}$, and a new price $p'_i > p_i$ such that $D(\mathbf{p}) = \{\mathbf{x}\}$ and $x'_j > x_j$ (resp. $x'_j < x_j$).

By negating $\mathbf{d}$ if necessary, we can assume that $d_i < 0$, so $d_j > 0$ (resp. $d_j < 0$) must hold. Let $X = \{\mathbf{0}, \mathbf{d}\}$ and define a valuation $V : X \to \mathbb{R}$ by $V(\mathbf{x}) = 0$. As shown in the proof of Proposition D.1, valuation V is of demand type $\mathcal{D}$, and there exists $\lambda > 0$ such that letting $\mathbf{p} = ((-\lambda)_i, \mathbf{0}_{I \smallsetminus \{i\}})$ and $p'_i = \lambda$, we have that $D(\mathbf{p}) = \{\mathbf{0}\}$ and $D(p'_i, \mathbf{p}_{I \smallsetminus \{i\}}) = \{\mathbf{d}\}$. To complete the proof, we simply note that $d_j > 0$ (resp. $d_j < 0$) holds by our normalization of $\mathbf{d}$. □

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