Proof that the Strong Substitutes Product-Mix Auction Bidding Language can represent any Strong Substitutes preferences

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February 12, 2021

Very preliminary draft under revision
Please send corrections and questions

Abstract

This note proves that all strong-substitutes preferences can be represented by appropriate sets of bids of the kind permitted by the Strong Substitutes Product-Mix Auction Bidding Language.

It is a draft of material that will form part of "Implementing Walrasian Equilibrium—the Language of Product-Mix Auctions" (Baldwin and Klemperer, in preparation).

1 Introduction

This note contains material, a revised version of which will form part of "Implementing Walrasian Equilibrium—the language of Product-Mix Auctions" (Baldwin and Klemperer, in preparation), which paper develops the theoretical underpinnings for, and further extensions of, the Product-Mix Auction design that was developed for the Bank of England in 2007-8.

Specifically, this note shows that the strong-substitutes product-mix auction (SSPMA) bidding language permits the specification of precisely the set of preferences that are strong substitutes.[1] That is, all strong-substitutes preferences can be represented by appropriate sets of bids of the kind permitted by the SSPMA. Moreover, no permitted combination of SSPMA bids represents any other form of preferences.[2] These results are not only important for the usefulness of the auction; the results are also significant because the SSPMA language is the only language we know to have these properties.

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[1] Baldwin, Goldberg, and Klemperer (2016b) provided a sketch of the proof given here; Klemperer (2010) stated the result for the case of multiple units of each of two goods.

[2] Permitted or "valid" combinations of SSPMA bids are such that the demand for a good cannot decrease if its price falls while no other price changes. See Section 4.2 and Baldwin, Goldberg, Klemperer, and Lock (2019), Theorem 1, and Baldwin, Goldberg, and Klemperer (2016a) for more detail.

By contrast, for example, neither Hatfield and Milgrom (2005)'s endowed assignment messages nor Milgrom (2009)'s (integer) assignment messages can express all strong substitute valuations (see Ostrovsky and Paes Leme 2015, and Fichtl 2020, respectively).[3]

Strong-substitutes preferences are those that would be ordinary-substitutes preferences if we treated every unit of every good as a separate good. Such preferences have many attractive theoretical properties; they mean, for example, that if the price of any one good increases, and the demand for it decreases, then the demand for all other goods can increase by at most the amount of that decrease.[4] Such preferences also naturally arise in practical applications. For example, bidders' preferences in liquidity auctions run by the Bank of England seem to be well-represented by these preferences. The SSPMA was therefore developed by Klemperer (2008) for the Bank of England, which implemented a simplified version of it. See Klemperer (2018) for discussion of variants of the language, and see Baldwin et al. (2019), Baldwin, Bichler, Fichtl, and Klemperer (2021) for algorithms to solve the SSPMA, i.e., find competitive equilibrium prices and allocations, given any valid sets of bids.[5]

We proceed as follows. Section 2 gives first definitions and conventions, including explaining the SSPMA bidding language, and states the main results. Section 3 provides further background technical definitions and develops results that are needed in the proof, including developing and "arithmetic" on the "locus of indifference prices (LIPs)" that we introduced in Baldwin and Klemperer (2019). Section 4 returns to the SSPMA bidding language and re-interprets it in terms of the material that has been introduced in Section 3. Section 5 provides the structure of the proofs, and some of the details, although other parts of the technical proofs are given in the Appendix.

2 Conventions, Definitions and Summary of Results

2.1 Valuations and Substitutes

This paper concerns the representation of strong substitutes preferences for indivisible goods, when utility is quasilinear.

That is, an agent has a real-valued *valuation* v on a finite *domain* $A \subseteq \mathbb{Z}^n_{\geq 0}$ of n indivisible *goods*, and so $v : A \to \mathbb{R}$ is a function.We write $[n] = \{1, \ldots, n\}$ for the set of all of these goods, and write $[n]_0$ for the set $\{0, 1, \ldots, n\}$.

Of particular importance are cases when the domain is a *simplex*. For $S \subseteq [n]_0$ write $\Delta_S := \{\mathbf{x} \in \mathbb{Z}^n_{\geq 0} \mid \sum_{i \in S} x_i \leq 1\}$ if $0 \in S$ and $\Delta_S := \{\mathbf{x} \in \mathbb{Z}^n_{\geq 0} \mid \sum_{i \in S} x_i = 1\}$ if $0 \notin S$. For $m \in \mathbb{Z}$ we slightly abuse notation by writing $m\Delta_S := \{m\mathbf{x} \in \mathbb{Z}^n \mid \mathbf{x} \in \mathrm{conv}\,(\Delta_S)\}$; note that we include the case $m < 0$ here.

Prices $\mathbf{p} \in \mathbb{R}^n$ are linear on these n goods, and there are no budget constraints, so that the agent's quasilinear utility takes the simple form $v(\mathbf{x}) - \mathbf{p} \cdot \mathbf{x}$ for all $\mathbf{x} \in A$. The agent demands any bundle that maximises its utility, so that its *demand set* at price $\mathbf{p}$ is $D_v(\mathbf{p}) = \arg\max_{\mathbf{x} \in A}\{v(\mathbf{x}) - \mathbf{p} \cdot \mathbf{x}\}$. Observe that the demand set need not be a single

[3]Furthermore, Tran (2020) shows that it is not possible to express all strong substitute valuations as combinations of weighted ranks of matroids on a ground set bounded by the number of goods.

[4]Strong substitutability is equivalent to $M^\natural$-concavity (see Murota and Shioura, 1999, Murota, 2003, Shioura and Tamura, 2015).

[5]Fichtl (2021) details an algorithm to solve budget-constrained PMAs.

bundle, although it will be at a dense set of prices in $\mathbb{R}^n$. Similarly, the *indirect utility* $u_v : \mathbb{R}^n \to \mathbb{R}$ is given by $\max_{\mathbf{x}\in A}\{v(\mathbf{x}) - \mathbf{p}\cdot\mathbf{x}\}$; we observe that this is always a convex function.

The valuations v considered in this paper will always be *concave*: the set A satisfies $\text{conv}(A) \cap \mathbb{Z}^n = A$, and for every $\mathbf{x} \in A$ there exists $\mathbf{p} \in \mathbb{R}^n$ such that $\mathbf{x} \in D_v(\mathbf{p})$. And they will always be *strong substitutes* valuations, in the following standard sense:

Definition 2.1 (See e.g. Ausubel and Milgrom (2002) and Milgrom and Strulovici (2009))**.** Let $v : A \to \mathbb{R}$ be a valuation.

(1) v is *ordinary substitutes* if, for any prices $\mathbf{p}' \geq \mathbf{p}$ with $D_v(\mathbf{p}) = \{\mathbf{x}\}$ and $D_v(\mathbf{p}') = \{\mathbf{x}'\}$, we have $x'_k \geq x_k$ for all k such that $p'_k = p_k$.
(2) v is *strong substitutes* if, when we consider every unit of every good to be a separate good, it is a valuation for ordinary substitutes.

Strong substitutes valuations are the natural extension of Kelso and Crawford (1982)'s "gross substitutes" to the multi-unit case, and are also known as "$M^\natural$-concave functions" in the literature on discrete convex analysis (See, e.g. Murota, 2003).

The simplest examples of strong substitutes valuations are the "unit demands" of Gul and Stacchetti (1999).

Example 2.2. A unit demand valuation with location $\mathbf{r}$ is the valuation v with domain $\Delta_{[n_0]} = \{\mathbf{x} \in \mathbb{Z}^n_{\geq 0} \mid \sum_{i\in[n]} x_i \leq 1\}$ and value $v(\sum_i c_i\mathbf{e}^i) = \sum_i c_i r_i$ for $\sum_i c_i \mathbf{e}^i \in \Delta_{[n_0]}$.

Note that the agent is indifferent between all the bundles $\mathbf{x} \in \Delta_{[n_0]}$ when $\mathbf{p} = \mathbf{r}$, as all deliver utility $v(\mathbf{x}) - \mathbf{p}\cdot\mathbf{x} = 0$ at that price.

This paper shows that all strong substitutes valuations can be built up from unit demand valuations.

2.2 SSPMA Bids and the Representation Theorems

To state our representation, we first give a quick definition of the bids in the SSPMA bidding language, which for brevity we will refer to just as "bids", from which we will build all strong substitutes valuations.

Definition 2.3. A *positive bid* $\mathbf{b} = (\mathbf{r}, m)$ with *location* $\mathbf{r} = \ell(\mathbf{b}) \in \mathbb{R}^n$ and *multiplicity* $m = m(\mathbf{b}) \in \mathbb{Z}_{>0}$ represents valuation $v_\mathbf{b}$ with domain $m\Delta_{[n_0]}$ and value $v_\mathbf{b}(\mathbf{x}) = \sum_i x_i r_i$ for $\mathbf{x} \in m\Delta_{[n_0]}$.

We write $D_\mathbf{b}(\mathbf{p}) := D_{v_\mathbf{b}}(\mathbf{p})$ to simplify notation. It is easy to see that a bid with multiplicity m is an aggregate of m identical unit demand valuations as in Example 2.2.

For convenience in referring to the case in which $\mathbf{0}$ is demanded, we write $\ell(\mathbf{b})_0 = 0$ and $p_0 = 0$. Observe now that the indirect utility $u_\mathbf{b}$ of $v_\mathbf{b}$ is:

$$u_\mathbf{b}(\mathbf{p}) = m(\mathbf{b}) \max_{i\in[n]_0}(\ell(\mathbf{b})_i - p_i). \tag{1}$$

and so:

$$D_\mathbf{b}(\mathbf{p}) = \left\{ \mathbf{x} \in m(\mathbf{b})\Delta_{[n_0]} \mid \sum_{i=1}^{n} x_i(r_i - p_i) = u_\mathbf{b}(\mathbf{p}) \right\} = m(\mathbf{b})\Delta_{\arg\max_{i\in[n]_0}(\ell(\mathbf{b})_i - p_i)}. \tag{2}$$

Given a finite multiset $\mathcal{B}$ of positive bids, aggregate demand is $D_{\mathcal{B}}(\mathbf{p}) = \sum_{\mathbf{b}\in\mathcal{B}} D_{\mathbf{b}}(\mathbf{p})$ and aggregate indirect utility is $u_{\mathcal{B}}(\mathbf{p}) = \sum_{\mathbf{b}\in\mathcal{B}} u_{\mathbf{b}}(\mathbf{p})$. If, for every strong substitutes valuation v, there were to exist a finite multiset $\mathcal{B}$ of positive bids such that $D_v(\mathbf{p}) = D_{\mathcal{B}}(\mathbf{p})$ for all $\mathbf{p} \in \mathbb{R}^n$, then our paper might end here. But this is not possible. So we introduce *negative bids*: those for which $m(\mathbf{b}) \in \mathbb{Z}_{<0}$.

A negative bid $\mathbf{b}$ does not correspond to a valuation $v_{\mathbf{b}}$. However, we may, none the less, define $u_{\mathbf{b}}$ and $D_{\mathbf{b}}(\mathbf{p})$ for such a bid as in Equations (1) and (2) above. Write, for convenience, $|\mathbf{b}| = (\ell(\mathbf{b}), |m(\mathbf{b})|)$, and observe that if $m(\mathbf{b}) < 0$ then

$$D_{\mathbf{b}}(\mathbf{p}) = -D_{|\mathbf{b}|}(\mathbf{p}) = \{-\mathbf{x} \mid \mathbf{x} \in D_{|\mathbf{b}|}(\mathbf{p})\}. \tag{3}$$

An increase in the price of good i leads to a weak *increase* in demand for good i, and so $D_{\mathbf{b}}$ does not in this case correspond to a quasilinear demand correspondence.

If $\mathcal{B}$ is a finite multiset of bids (of either sign), we define the indirect utility $u_{\mathcal{B}}$ via $u_{\mathcal{B}}(\mathbf{p}) = \sum_{\mathbf{b}\in\mathcal{B}} u_{\mathcal{B}}(\mathbf{p})$. Recalling that the indirect utility of a valuation is convex, we define the bids to be *valid* if $u_{\mathcal{B}}$ is convex. However, assigning a demand correspondence to bids $\mathcal{B}$ is a little more delicate. First observe that, for each $\mathbf{b} \in \mathcal{B}$, demand $D_{\mathbf{b}}(\mathbf{p})$ is single-valued at a dense set of prices in $\mathbb{R}^n$. So, since there are finitely many $\mathbf{b} \in \mathcal{B}$, we may identify the set Q of all price vectors $\mathbf{q}$ in a small open neighbourhood of $\mathbf{p}$, such that $D_{\mathbf{b}}(\mathbf{p})$ is single-valued for all $\mathbf{q} \in Q$. Now define

$$D_{\mathcal{B}}(\mathbf{p}) := \operatorname{conv}\left\{\sum_{\mathbf{b}\in\mathcal{B}} D_{\mathbf{b}}(\mathbf{q}) \mid \mathbf{q} \in Q\right\} \cap \mathbb{Z}^n \tag{4}$$

Baldwin et al. (2019) show the following result.

Fact 2.4 (Baldwin et al. 2019, Theorem 1)**.** Let $\mathcal{B}$ be a finite multiset of bids, and $u_{\mathcal{B}}$ the associated indirect utility function. The following are equivalent:

(1) $\mathcal{B}$ is valid.
(2) For every $\mathbf{p} \in \mathbb{R}^n$ and every $i, j \in [n]_0$ with $i \neq j$, the set $\mathcal{B}'$ of bids $\mathbf{b} \in \mathcal{B}$ such that $i, j \in \arg\max_{i'\in[n]_0}(\ell(\mathbf{b})_{i'} - p_{i'})$ satisfies $\sum_{\mathbf{b}\in\mathcal{B}'} m(\mathbf{b}) \geq 0$.
(3) $u_{\mathcal{B}}$ is the indirect utility function of a strong substitutes valuation v such that $D_v(\mathbf{p}) = D_{\mathcal{B}}(\mathbf{p})$ for all $\mathbf{p} \in \mathbb{R}^n$.

And the domain of the strong substitutes valuation corresponding to $\mathcal{B}$ is a simplex:

Fact 2.5 (Baldwin et al. (2021) Proposition 3)**.** Let $\mathcal{B}$ be a finite valid multiset of bids, let $A = \bigcup\{D_{\mathcal{B}}(\mathbf{p}) \mid \mathbf{p} \in \mathbb{R}^n\}$, and let $W := \sum_{\mathbf{b}\in\mathcal{B}} m(\mathbf{b})$. Then $A = W\Delta_{[n]_0}$.

Thus it will be most straightforward to provide our representation for valuations whose domain is a simplex. And our central result is that the SSPMA bidding language here described can indeed represent all such valuations:

Theorem 2.6. *If v is a strong substitutes valuation with domain $W\Delta_{[n]_0}$ for some $W \in \mathbb{Z}_{\geq 0}$ then there exists a valid bid collection $\mathcal{B}$ such that $v_{\mathcal{B}} = v$.*

If the domain of v is not a simplex, Fact 2.5 shows us that we cannot represent v using a set of bids $\mathcal{B}$ globally, i.e. for all $\mathbf{p} \in \mathbb{R}^n$. However, we can do so in any sufficiently large bounded region of $\mathbb{R}^n$, as we show. For any $H \gg 0$ we write $\mathbf{H} := [-H, H]^n$ so that $\mathbf{H}^\circ = (-H, H)^n$.

Theorem 2.7. *If v is a strong substitutes valuation then for any $H \gg 0$ there exists a valid bid collection $\mathcal{B}$ such that $\ell(\mathbf{b}) \in \mathbf{H}$ for all $\mathbf{b} \in \mathcal{B}$ and such that $D_v(\mathbf{p}) = D_{\mathcal{B}}(\mathbf{p})$ for all $\mathbf{p} \in \mathbf{H}^\circ$.*

Thus, by taking very large values of H, we can present the demand set at all prices that might arise in practice. This result rests on our proof that any strong substitutes valuation can be presented in a large enough bounded subset of $\mathbb{R}^n$ as a valuation whose domain is a simplex.

To extend the representation of Theorem 2.7 to the whole of $\mathbb{R}^n$, we need to modify the interpretation of the bids of Theorem 2.7 lying on the boundary $\partial\mathbf{H} := \mathbf{H} \setminus \mathbf{H}^\circ$. Now write, for any $\mathbf{b} \in \mathcal{B}$, the demand set

$$D_{\mathbf{b}}^{\mathbf{H}}(\mathbf{p}) := D_{\mathbf{b}}(\mathbf{p}) \cap m(\mathbf{b})\Delta_{S(\ell(\mathbf{b}),\mathbf{H})}$$

where

$$S(\ell(\mathbf{b}),\mathbf{H}) = \begin{cases} \{i \in [n] \mid \ell(\mathbf{b})_i \neq -H\} \cup \{0\} & \text{if } \ell(\mathbf{b})_j \neq H \text{ for all } j \in [n] \\ \{i \in [n] \mid \ell(\mathbf{b})_i \neq -H\} & \text{if } \ell(\mathbf{b})_j = H \text{ for some } j \in [n]. \end{cases}$$

We will see later that $D_{\mathbf{b}}^{\mathbf{H}}(\mathbf{p}) = D_{\mathbf{b}}(\mathbf{p})$ for all $\mathbf{p} \in \mathbf{H}^\circ$, and that $D_{\mathbf{b}}^{\mathbf{H}}(\mathbf{p}) = D_{\mathbf{b}}(\mathbf{p})$ for all $\mathbf{p} \in \mathbb{R}^n$ if $\ell(\mathbf{b}) \in \mathbf{H}^\circ$.[6] We can now define $D_{\mathcal{B}}^{\mathbf{H}}(\mathbf{p})$ in the same way as we did in and above Equation (4), and we show

Theorem 2.8. *If v is a strong substitutes valuation then for sufficiently large $H \gg 0$ there exists a valid bid collection $\mathcal{B}$ such that $\ell(\mathbf{b}) \in \mathbf{H}$ for all $\mathbf{b} \in \mathcal{B}$ and such that $D_v(\mathbf{p}) = D_{\mathcal{B}}^{\mathbf{H}}(\mathbf{p})$ for all $\mathbf{p} \in \mathbb{R}^n$.*

Finally, we address uniqueness of the valid bid collections found by Theorems 2.6–2.8.

Theorem 2.9. *For each of Theorems 2.6–2.8, there is a unique valid bid collection as described if we restrict attention to valid bid collections with at most one bid at any location in $\mathbb{R}^n$; and there is an unique valid bid collection as described if we restrict attention to valid bid collections for which each bid has multiplicity ± 1, and a bid with multiplicity $+1$ may not be in the same location as a bid with multiplicity -1.*

This result illustrates that the form uniqueness takes, depends on one's choice of convention. One can allow bids of any (non-zero weight), but allow no two bids to be placed in the same location. We use this convention in Baldwin et al. (2021). Or one can restrict bid multiplicities to be only ± 1. We use this convention in Baldwin et al. (2019). There is an obvious one-to-one correspondence between these two conventions, and one may use either depending on convenience.

3 Background Technical Material

To prove these results, we first need to develop some technical machinery.

[6] See Lemma 4.12.

3.1 Key definitions and results relating to LIPs

We need to record several definitions from Baldwin and Klemperer (2019), first relating to the "LIP"; we introduced these from the literature on "tropical geometry".

Definition 3.1 (Baldwin and Klemperer 2019 Definitions 2.1, 2.2 and 2.3; see also e.g. Maclagan and Sturmfels 2015)**.** Let $v : A \to \mathbb{R}$ be a valuation on a finite set of bundles $A \subseteq \mathbb{Z}^n$.

(1) The *Locus of Indifference Prices (LIP)* is $\mathcal{L}_v := \{\mathbf{p} \in \mathbb{R}^n \mid |D_v(\mathbf{p})| > 1\}$.
(2) A *unique demand region (UDR)* of a valuation v is the set of all prices at which a given bundle in A is uniquely demanded. That is, it has the form $\{\mathbf{p} \in \mathbb{R}^n : \{\mathbf{x}\} = D_v(\mathbf{p})\}$ for some $\mathbf{x} \in A$.
(3) A *facet* of $\mathcal{L}_v$ is a subset $F \subseteq \mathcal{L}_v$ such that there exist $\mathbf{x}^1, \mathbf{x}^2 \in A$, with $\mathbf{x}^1 \neq \mathbf{x}^2$, satisfying $F = \{\mathbf{p} \in \mathcal{L}_v \mid \mathbf{x}^1, \mathbf{x}^2 \in D_v(\mathbf{p})\}$ and such that $\dim F = n - 1$.[7]
(4) Let $\mathbf{x}, \mathbf{x}'$ be the bundles demanded in the UDRs on either side of facet F. The *weight* of F, $w_v(F)$, is the greatest common divisor of the entries of $\mathbf{x}' - \mathbf{x}$.
(5) A *price complex cell* of v is a non-empty set $C \subseteq \mathbb{R}^n$ such that there exist $\mathbf{x}^1, \ldots, \mathbf{x}^k \in A$, with $k \geq 1$, satisfying $C = \{\mathbf{p} \in \mathbb{R}^n \mid \mathbf{x}^1, \ldots, \mathbf{x}^k \in D_v(\mathbf{p})\}$.
(6) The *price complex* is the set of all price complex cells.
(7) The *cells of the LIP* are the price complex cells contained in the LIP.

Particularly important is the relationship between facet normals and changes in demand:

Fact 3.2 (Baldwin and Klemperer 2019 Proposition 2.4)**.**

(1) If $\mathbf{x}$, $\mathbf{x}'$ are uniquely demanded on either side of facet F, then $\mathbf{p} \cdot (\mathbf{x}' - \mathbf{x})$ is constant for all $\mathbf{p} \in F$.
(2) The change in demand as price changes between the UDRs on either side of F, is $w_u(F)$ times the primitive integer vector that is normal to F, and that points in the opposite direction to the change in price.

Our second collection of recalled definitions relate to polyhedral complexes. A modification made here, relative to Baldwin and Klemperer (2019), is to allow negative weightings.

Definition 3.3.

(1) A *rational polyhedron* is the intersection of a finite set of half-spaces $\{\mathbf{p} \in \mathbb{R}^n \mid \mathbf{p} \cdot \mathbf{d} \leq \alpha\}$ for some $\mathbf{d} \in \mathbb{Z}^n$ and $\alpha \in \mathbb{R}$.
(2) A *face* of a polyhedron C maximises $\mathbf{p} \cdot \mathbf{d}$ over $\mathbf{p} \in C$, for some fixed $\mathbf{d} \in \mathbb{R}^n$.
(3) The *interior* of polyhedron C is $C^\circ := \{\mathbf{p} \in C \mid \mathbf{p} \notin C' \text{ for any face } C' \subsetneq C\}$.
(4) A *rational polyhedral complex* Π is a finite collection of *cells* $C \subseteq \mathbb{R}^n$ such that:
    (i) if $C \in \Pi$ then C is a rational polyhedron and any face of C is also in Π;
    (ii) if $C, C' \in \Pi$ then either $C \cap C' = \emptyset$ or $C \cap C'$ is a face of both C and C'.
(5) A *k-cell* is a cell of dimension k. A *facet* is a cell of dimension $n - 1$.
(6) A polyhedral complex is *k-dimensional* if all its cells are contained in its k-cells.

[7]The dimension of a set $F \subseteq \mathbb{R}^n$ is the dimension of its affine span, i.e. the dimension of the smallest linear subspace $U \subseteq \mathbb{R}^n$ such that $F \subseteq \{\mathbf{c}\} + U$ for some fixed vector $\mathbf{c}$.

(7) A $\mathbb{Z}$-*weighted* polyhedral complex is a pair $(\Pi, \mathbf{w})$ where Π is a polyhedral complex and $\mathbf{w}$ is a vector assigning a weight $w(F) \in \mathbb{Z}$ to each facet $F \in \Pi$.
(8) A $\mathbb{Z}_{>0}$-*weighted* polyhedral complex $\mathbb{Z}$-weighted polyhedral complex in which $w(F) > 0$ for all facets $F \in \Pi$.
(9) An $(n-1)$-dimensional $\mathbb{Z}_{>0}$-weighted rational polyhedral complex Π is *balanced* if, for every $(n-2)$-cell $G \in \Pi$, the weights $w(F^j)$ on the facets $F^1 \ldots F^l$ that contain G, and primitive integer normal vectors $\mathbf{d}_{F^j}$ for these facets that are defined by a fixed rotational direction about G, satisfy $\sum_{j=1}^{l} w(F^j)\mathbf{d}_{F^j} = 0$.

The notions from Definition 3.1 are related to those from Definition 3.3 via the following two results:

Fact 3.4 (See, e.g. Baldwin and Klemperer (2019) Proposition 2.7)**.**

(1) The price complex is an n-dimensional rational polyhedral complex.
(2) The LIP cells, paired with the facet weights, form an $(n-1)$-dimensional $\mathbb{Z}_{>0}$-weighted rational polyhedral complex.

Fact 3.5 (**The Valuation-Complex Equivalence Theorem**, Mikhalkin (2004), Remark 2.3 and Prop. 2.4; Baldwin and Klemperer (2019) Theorem 2.14)**.** Suppose that $(\Pi, \mathbf{w})$ is an $(n-1)$-dimensional $\mathbb{Z}_{>0}$-weighted rational polyhedral complex in $\mathbb{R}^n$, that $\mathcal{L}$ is the union of the cells in Π, and $\mathbf{p}$ is any price not contained in $\mathcal{L}$.

(1) There exists a finite set $A \subsetneq \mathbb{Z}^n$ and a function $u : A \to \mathbb{R}$ such that $\mathcal{L}_v = \mathcal{L}$ and $\mathbf{w}_u = \mathbf{w}$, if and only if $(\Pi, \mathbf{w})$ is balanced.
(2) If $(\Pi, \mathbf{w})$ is balanced then there exists a finite set $A \subsetneq \mathbb{Z}^n$ and a unique *concave* valuation $u : A \to \mathbb{R}$ such that $D_v(\mathbf{p}) = \{\mathbf{0}\}$, $u(\mathbf{0}) = 0$, $\mathcal{L}_v = \mathcal{L}$ and $\mathbf{w}_u = \mathbf{w}$.

Baldwin and Klemperer (2019) introduced "demand type" to classify economic properties of valuations via the shapes of their facets. First, *demand type vector sets* $\mathcal{D} \subseteq \mathbb{Z}^n$ consist of primitive integer vectors and satisfy $\mathbf{d} \in \mathcal{D} \Rightarrow -\mathbf{d} \in \mathcal{D}$. The, for any demand type vector set, the *demand type* is the set of valuations v such that every facet of $\mathcal{L}_v$ has normal vector in $\mathcal{D}$.

The case of interest for this paper is the *strong substitutes demand type vector set*, given in dimension n by $\{\mathbf{e}^i, \mathbf{e}^i - \mathbf{e}^j \mid i, j \in [n], i \neq j\}$. A valuation is strong substitutes if and only if it is concave and of the strong substitutes demand type (Baldwin and Klemperer, 2014, Shioura and Tamura, 2015). Therefore its facets must take one of two simple forms, and we introduce the following terminology:

Definition 3.6.

(1) For $i \in [n]$, a facet of $\mathcal{L}_v$ is a *i-hod* if it has normal vector $\mathbf{e}^i$. It is a *hod* if it is an i-hod for some $i \in [n]$.
(2) For $i \in [n]$ with $i \neq j$, a facet of $\mathcal{L}_v$ is a *ij-fin* if it has normal vector $\mathbf{e}^i - \mathbf{e}^j$. It is a *fin* if it is an ij-fin for some $i, j \in [n]$ with $i \neq j$.

We use these terms because, when $n = 3$, the "hods" appear to form a builders' hod, and the fins resemble the fins or blades of a turbine.

3.2 Orders for the Facets of a Strong Substitutes LIP

It is well-known that $(\mathbb{R}^n, \leq)$ is a lattice in the order-theoretic sense, where $\leq$ is the "Euclidean" (partial) ordering (see, for example Topkis, 1978). We review the definition of $\leq$, and provide additional orderings on subsets of $\mathbb{R}^n$ that are affine spans of facets of strong substitutes LIPs. These will be important for identifying key points on facets, which will be used to understand when our "bids" can "cover" these facets.

Definition 3.7.

(1) Define $\leq$ on $\mathbb{R}^n$ by $\mathbf{x} \leq \mathbf{y}$ if $x_i \leq y_i$ for all $i \in [n]$.
(2) For any $i \in [n]$ define $\leq_i$ on $\mathbb{R}^n$ by $\mathbf{x} \leq_i \mathbf{y}$ if $x_i = y_i$ and $x_k \leq y_k$ for all $k \in [n]$.
(3) For any $i, j \in [n]$ with $i \neq j$ define $\leq_{ij}$ on $\mathbb{R}^n$ by $\mathbf{x} \leq_{ij} \mathbf{y}$ if $x_i - x_j = y_i - y_j$ and $x_i \geq y_i$ (equivalently $x_j \geq y_j$) and $x_k - x_i \leq y_k - y_i$ for all $k \in [n]$ with $k \neq i, j$.

It may appear that the condition on coordinates i and j in $\leq_{ij}$ is the "wrong way around". We choose this convention so that a bid is the minimal point with respect to a suitable ordering, of all the facets associated with it, as will be made clear below (Corollary 4.3).

These orders do give rise to lattices, as we show:

Lemma 3.8. *For any $i, j \in [n]$ with $i \neq j$ and any $\mathbf{r} \in \mathbb{R}^n$, we have:*

(1) $(\{\mathbf{p} \in \mathbb{R}^n \mid p_i = r_i\}, \leq_i)$ *is a lattice in the order-theoretic sense;*
(2) $(\{\mathbf{p} \in \mathbb{R}^n \mid p_i - p_j = r_i - r_j\}, \leq_{ij})$ *is a lattice in the order-theoretic sense.*

This allows us to write:

Definition 3.9. For any $i, j \in [n]$ with $i \neq j$ and any $\mathbf{r} \in \mathbb{R}^n$:

(1) if $F \subseteq \{\mathbf{p} \in \mathbb{R}^n \mid p_i = r_i\}$, write $\bigwedge_i F$ for the infimum of F with respect to $\leq_i$;
(2) if $F \subseteq \{\mathbf{p} \in \mathbb{R}^n \mid p_i - p_j = r_i - r_j\}$, write $\bigwedge_{ij} F$ for the infimum of F with respect to $\leq_{ij}$.

It is well-known that, with strong substitutes valuations, the set of competitive equilibrium prices forms a lattice with respect to $\leq$. These new orders allow us to give analogous results for every facet of a strong substitutes LIP.

Lemma 3.10. *Suppose $\mathcal{L}_v$ is a strong substitutes LIP.*

(1) If F is an i-hod of $\mathcal{L}_v$ which is bounded below with respect to $\leq_i$ then $\bigwedge_i F \in F$
(2) If F is an ij-fin of $\mathcal{L}_v$ which is bounded below with respect to $\leq_{ij}$ then $\bigwedge_{ij} F \in F$.

A particularly nice feature of valuations with simplex domain, is that the bounds of Lemma 3.10 always exist:

Corollary 3.11. *Suppose $v : A \to \mathbb{R}$ is a strong substitutes valuation with domain $A = D\Delta_{[n_0]}$ for some $D \in \mathbb{Z}_{\geq 0}$.*

(1) If F is an i-hod of $\mathcal{L}_v$ then $\bigwedge_i F \in F$
(2) If F is an ij-fin of $\mathcal{L}_v$ then $\bigwedge_{ij} F \in F$.

3.3 Pseudo-LIPs and their Arithmetic

Recall, again from Baldwin and Klemperer (2019) that, when we have multiple valuations from multiple agents:

Definition 3.12 (Baldwin and Klemperer (2019) Definition 3.12)**.** An *aggregate valuation* of $\{v^j \mid j \in J\}$ is a valuation v^J with domain $A := \sum_{j\in J} A^j$ such that $D_{v^J}(\mathbf{p}) = \sum_{j\in J} D_{v^j}(\mathbf{p})\ \forall \mathbf{p} \in \mathbb{R}^n$.

Fact 3.13 (Baldwin and Klemperer (2019) Lemma 3.13)**.** Given a finite set of valuations $\{v^j \mid j \in J\}$:

(1) an aggregate valuation v^J exists;

(2) $\mathcal{L}_{v^J} = \bigcup_{j\in J} \mathcal{L}_{v^j}$;

(3) If F is a facet of $\mathcal{L}_{v^J}$, then $w_{v^J}(F) = \sum_{F^j\in\mathcal{F}} w_{v^j}(F^j)$, in which $\mathcal{F}$ is the set of all facets of the individual $\mathcal{L}_{v^j}$ which contain F.

Here we introduce a new way to think about this aggregation: as addition of polyhedral complexes. We will write $(\mathcal{L}_{v^1}, w_{v^1}) \boxplus (\mathcal{L}_{v^2}, w_{v^2})$ for the weighted LIP of the aggregate valuation $v^{\{1,2\}}$ as defined above. We will also introduce an analogous "subtraction", which we will notate $\boxminus$. This is to allow us to use our "negative bids". As we explained in Section 2.2, if $m(\mathbf{b}) < 0$ then bid $\mathbf{b}$ does not correspond to a meaningful economic valuation. But, considering our definition of their demand sets (Equation (3)) and the properties of weights of facets of LIPs (Definition 3.1 part (4) and Fact 3.2), it is natural to associate this bid with the set $\mathcal{L}_{|\mathbf{b}|}$ but to weight each facet with the negative number $m(\mathbf{b})$.[8] We shall indeed do. However, this is not a true "location of indifference prices", and so we must widen our class of objects of study, as follows:

Definition 3.14. Fix (Π, w), where Π is a rational polyhedral complex of dimension n, such that $\bigcup \Pi = \mathbb{R}^n$, and such that w is a balanced $\mathbb{Z}$-weighting on the facets (that is, the $(n-1)$-dimensional cells) of Π.

The *weighted pseudo-LIP* of (Π, w) is the pair $(\mathcal{L}, w)$, where $\mathcal{L}$ is the union of the facets F of Π such that $w(F) \neq 0$, and the weight w on facets $F \subseteq \mathcal{L}$ is inherited from (Π, w).

Because we only define a pseudo-LIP of a balanced complex (Π, w), it follows that every pseudo-LIP is balanced.

The facets of a pseudo-LIP $\mathcal{L}$ are inherited from Π. Unlike the case of a (true) LIP, they are not necessarily the maximal $(n-1)$-dimensional linear pieces that only intersect in their boundaries; for example, two facets in Π, with the same affine span, could meet in Π along an $(n-2)$-cell at which the only other facets present have weight 0; one might wish to merge these facets in $\mathcal{L}$. We shall not explicitly do so, because doing so could give rise to non-convex "facets". However, we are not too concerned with the subdivision of the maximal $(n-1)$-dimensional linear pieces of $\mathcal{L}$ into facets, because:

Lemma 3.15. *If F, F' are facets of pseudo-LIP $(\mathcal{L}, w)$ such that $F \cap F'$ is $(n-2)$-dimensional and no other facets of $(\mathcal{L}, w)$ contain $F \cap F'$ in their boundary, then F and F' share a common affine span and $w(F) = w(F')$.*

[8] Recall that $|\mathbf{b}| := (\ell(\mathbf{b}), |m(\mathbf{b})|)$.

Proof. The balancing condition holds around $F \cap F'$. So, if $\mathbf{d}$ and $\mathbf{d}'$ are the respective primitive integer normal vectors to F and F', chosen with respect to a coherent rotational direction, then $w(F)\mathbf{d} + w(F')\mathbf{d}' = \mathbf{0}$. Since $F \cap F'$ is $(n-2)$-dimensional, we reject $\mathbf{d}' = \mathbf{d}$, and so $\mathbf{d}' = -\mathbf{d}$, showing their affine spans are the same, and $w(F) = w(F')$. □

Therefore, if $\mathcal{L} = \mathcal{L}'$ and if $w(F) = w(F')$ whenever $F \cap F'$ is $(n-1)$-dimensional for a facet F of $\mathcal{L}$ and a F' of $\mathcal{L}'$, we shall abuse notation and say that $(\mathcal{L}, w) = (\mathcal{L}', w')$.

A positive-weighted pseudo-LIP is a "true" LIP of a valuation:

Proposition 3.16. *$(\mathcal{L}, w)$ is a weighted pseudo-LIP such that $w(F) > 0$ for all facets F of $\mathcal{L}$, if and only if $(\mathcal{L}, w) = (\mathcal{L}_v, w_v)$ for some valuation v.*

Proof. If v is a valuation then the price complex is a rational polyhedral complex of dimension n, the union of whose cells is $\mathbb{R}^n$, and the induced weighting w_v on the facets is positive and balanced. Thus $(\mathcal{L}_v, w_v)$ is a weighted pseudo-LIP. Conversely, let (Π, w) be the underlying complex defining $(\mathcal{L}, w)$. Observe that the set $\widehat{\Pi}$ of all facets F of Π such that $w(F) > 0$, taken together with all faces of these facets, is a rational polyhedral complex of dimension $(n-1)$. Restricting the weight w to the facets of $\widehat{\Pi}$, we see that the balancing condition is still satisfied, as all facets in Π that are not in $\widehat{\Pi}$ have weight 0. We can now invoke the Valuation-Complex Equivalence Theorem (Fact 3.5 above) to see that $(\mathcal{L}, w)$ is the LIP of a valuation v which induces the same weighting. □

However, in general we allow facets of pseudo-LIPs to have negative weight, because we are interested in subtraction as well as addition, as follows:

Definition 3.17. Let $(\mathcal{L}^1, w^1)$ and $(\mathcal{L}^2, w^2)$ be pseudo-LIPs.

(1) Define $(\mathcal{L}^1, w^1) \boxplus (\mathcal{L}^2, w^2)$ to be the pseudo-LIP $(\mathcal{L}, w)$ where
  (i) $\mathcal{L}$ is the closure of points $\mathbf{p}$ such that either $\mathbf{p}$ is in the interior of a facet F^1 of $\mathcal{L}^1$, or $\mathbf{p}$ is in the interior of a facet F^2 of $\mathcal{L}^2$, or both; and if indeed both hold then $w^1(F^1) + w^2(F^2) \neq 0$.
  (ii) If F is the facet of $\mathcal{L}$ containing $\mathbf{p}$ as in (1)(i), then set $w(F) = w^1_F + w^2_F$, where for $i = 1, 2$ we write $w^i_F = w^i(F^i)$ if F^i as described in (1)(i) exists, and we write $w^i_F = 0$ otherwise.

(2) Define $(\mathcal{L}^1, w^1) \boxminus (\mathcal{L}^2, w^2) := (\mathcal{L}^1, w^1) \boxplus (\mathcal{L}^2, -w^2)$, where if (Π^2, w^2) is the polyhedral complex defining $(\mathcal{L}^2, w^2)$, then $(\mathcal{L}^2, -w^2)$ is the pseudo-LIP defined by $(\Pi^2, -w^2)$.

If $(\mathcal{L}^i, w^i)$ is a (positive-weighted) LIP for $i = 1, 2$, then $(\mathcal{L}^1, w^1) \boxplus (\mathcal{L}^2, w^2)$ is identical to the aggregation described in Fact 3.13 above. Otherwise, the set of $(\mathcal{L}^1, w^1) \boxplus (\mathcal{L}^2, w^2)$ is formed by first applying the description of Fact 3.13 and then removing 0-weighted facets. We need to remove these to obtain our decompositions of strong substitutes LIPs. For recall, from Definition 3.1 Part (4) and Fact 3.2, that the weight of a facet is the greatest common divisor of the coordinate entries of the change in demand between UDRs on either side of this facet. If this weight is zero then the same bundle is demanded uniquely on either side, and hence in the interior of the facet itself: the facet should not be there.

Once addition "$\boxplus$" of $\mathbb{Z}$-weighted pseudo-LIPs is understood, the subtraction operation "$\boxminus$" is clear, as it is simply addition of the pseudo-LIP whose facets have the opposite sign.

We see that $\boxplus$ and $\boxminus$ have the following standard properties, which allow us to indeed think of them as an "arithmetic" of pseudo-LIPs.

Lemma 3.18. *If $(\mathcal{L}^1, w^1)$, $(\mathcal{L}^2, w^2)$ and $(\mathcal{L}^3, w^3)$ are weighted pseudo-LIPs, then so are $(\mathcal{L}^1, w^1)\boxplus(\mathcal{L}^2, w^2)$ and $(\mathcal{L}^1, w^1)\boxminus(\mathcal{L}^2, w^2)$, and the usual rules of addition and subtraction hold, with $(\emptyset, 0)$ playing the role of the identity element. That is:*

(1) $(\mathcal{L}^1, w^1) \boxplus (\mathcal{L}^2, w^2) = (\mathcal{L}^1, w^2) \boxplus (\mathcal{L}^1, w^1)$;
(2) $(\mathcal{L}^1, w^1) \boxplus ((\mathcal{L}^1, w^2) \boxplus (\mathcal{L}^3, w^3)) = ((\mathcal{L}^1, w^1) \boxplus (\mathcal{L}^1, w^2)) \boxplus (\mathcal{L}^3, w^3)$;
(3) $(\emptyset, 0) \boxplus (\mathcal{L}^1, w^1) = (\mathcal{L}^1, w^1) \boxplus (\emptyset, 0) = (\mathcal{L}^1, w^1)$;
(4) $(\mathcal{L}^1, w^1) \boxminus (\mathcal{L}^2, w^2) = (\emptyset, 0) \boxminus ((\mathcal{L}^2, w^2) \boxminus (\mathcal{L}^1, w^1))$;
(5) $(\mathcal{L}^1, w^1) \boxminus (\mathcal{L}^1, w^1) = (\emptyset, 0)$.

3.4 Covering

In general, even if $(\mathcal{L}^i, w^i)$ are (positive-weighted) LIPs for $i = 1, 2$, the pseudo-LIP $(\mathcal{L}^1, w^1) \boxminus (\mathcal{L}^2, w^2)$ need not be a LIP. However, as our objective is to construct LIPs themselves, we focus on the natural case in which this does hold, providing here a convenient way to refer to such cases.

Definition 3.19. If (X, w_X) and (Y, w_Y) are weighted $(n-1)$-dimensional polyhedral complexes we say that (Y, w_Y) *covers* (X, w_X) and write $(X, w_X) \preceq (Y, w_Y)$ if $X \subseteq Y$, and if $w_X(F) \le w_Y(F')$ for facets F of X and F' of Y, such that $F \cap F'$ is $(n-1)$-dimensional.

We phrase Definition 3.19 to apply more broadly than to pseudo-LIPs because in the proofs for our construction we will sometimes want to refer to, for example, a single weighted facet that is covered by a pseudo-LIP. However, considering LIPs themselves:

Lemma 3.20. *Suppose $(\mathcal{L}^i, w^i)$ are LIPs for $i = 1, 2$. Then $(\mathcal{L}^2, w^2) \preceq (\mathcal{L}^1, w^1)$ if and only if $(\mathcal{L}^1, w^1) \boxminus (\mathcal{L}^2, w^2)$ is a LIP.*

4 Relationship between Bids and pseudo-LIPs

4.1 The LIP from a Single Positive or Single Negative Bid

Recall, from Section 2.2, that a single positive bid $\mathbf{b} = (\mathbf{r}, m)$ with location $\ell(\mathbf{b}) = \mathbf{r}$ and multiplicity $m = m(\mathbf{b}) > 0$ represents the valuation $v_{\mathbf{b}}$ with domain $m\Delta_{[n_0]}$ and $v_{\mathbf{b}}(\mathbf{x}) = \sum_i x_i r_i$ for $\mathbf{x} \in m\Delta_{[n_0]}$. To simplify notation, we notate its weighted LIP $(\mathcal{L}_{\mathbf{b}}, w_{\mathbf{b}})$. And we can now define the corresponding object for a negative bid, whose demand set we recall from Equation (3):

Definition 4.1. The weighted pseudo-LIP $(\mathcal{L}_{\mathbf{b}}, w_{\mathbf{b}})$ of a bid $\mathbf{b}$ where $m(\mathbf{b}) < 0$ is defined by $\mathcal{L}_{\mathbf{b}} := \mathcal{L}_{|\mathbf{b}|}$ and $w_{\mathbf{b}}(F) = m(\mathbf{b})$ for all facets F of $\mathcal{L}_{\mathbf{b}}$.

As we show in the Appendix:

Lemma 4.2. *If $\mathbf{b} = (\mathbf{r}, m)$ then the weighted pseudo-LIP $(\mathcal{L}_{\mathbf{b}}, w_{\mathbf{b}})$ has weighted facets:*

(1) one i-hod for each $i \in [n]$, which we write $F^i_{\mathbf{b}}$, and such that:

(i) $F^i_{\mathbf{b}} = \{\mathbf{p} \in \mathbb{R}^n \mid p_i = r_i;\ p_j \geq r_j \text{ for } j \neq i\} = \{\mathbf{p} \in \mathbb{R}^n \mid \mathbf{r} \leq_i \mathbf{p}\}$;

(ii) $w_{\mathbf{b}}(F^i_{\mathbf{b}}) = m$;

(iii) *prices* $\mathbf{p}$ *are in* $F^i_{\mathbf{b}}$ *if and only if* $m\Delta_{\{0,i\}} \subseteq D_{\mathbf{b}}(\mathbf{p})$, *with equality for prices in the interior of* $F^i_{\mathbf{b}}$;

(2) *one* ij*-fin for each* $i, j \in [n]$ *with* $i \neq j$, *which we write* $F^{ij}_{\mathbf{b}}$, *and such that:*

(i) $F^{ij}_{\mathbf{b}} = \{\mathbf{p} \in \mathbb{R}^n \mid p_i \leq r_i,\ (p_i - r_i) = (p_j - r_j) \leq (p_k - r_k) \text{ for } k \neq i, j\}$
$= \{\mathbf{p} \in \mathbb{R}^n \mid \mathbf{r} \leq_{ij} \mathbf{p}\}$;

(ii) $w_{\mathbf{b}}(F^{ij}_{\mathbf{b}}) = m$;

(iii) *prices* $\mathbf{p}$ *are in* $F^i_{\mathbf{b}}$ *if and only if* $m\Delta_{\{i,j\}} \subseteq D_{\mathbf{b}}(\mathbf{p})$, *with equality for prices in the interior of* $F^{ij}_{\mathbf{b}}$.

It is immediate from Lemma 4.2 that:

Corollary 4.3. *Suppose* $\mathbf{b} = (\mathbf{r}, m)$ *is a bid and* v *is a strong substitutes valuation.*

(1) $\mathcal{L}_{\mathbf{b}}$ *is of the strong substitutes demand type.*
(2) $\bigwedge_i F^i_{\mathbf{b}} = \mathbf{r}$ *for all* $i \in [n]$.
(3) $\bigwedge_{ij} F^{ij}_{\mathbf{b}} = \mathbf{r}$ *for all* $i, j \in [n]$ *with* $i \neq j$.
(4) *If* F *is an* i*-hod of* $\mathcal{L}_v$ *for some* $i \in [n]$, *if* $\mathbf{r} \leq_i \bigwedge_i F$ *and if* $m \geq w_v(F)$ *then* $(F, w_v(F)) \preceq (\mathcal{L}_{\mathbf{b}}, w_{\mathbf{b}})$.
(5) *If* F *is an* ij*-fin of* $\mathcal{L}_v$ *for some* $i, j \in [n]$ *with* $i \neq j$, *if* $\mathbf{r} \leq_{ij} \bigwedge_i F$ *and if* $m \geq w_v(F)$ *then* $(F, w_v(F)) \preceq (\mathcal{L}_{\mathbf{b}}, w_{\mathbf{b}})$.

4.2 Aggregations of bids and identifying demand

We formally define our collections of bids, and the associated pseudo-LIPs:

Definition 4.4.

(1) A *bid collection* $\mathcal{B}$ is a finite multiset of bids $\mathbf{b} = (\mathbf{r}, m)$ where $\mathbf{r} \in \mathbb{R}^n$ and $m \in \mathbb{Z} \setminus \{0\}$.
(2) For a bid collection $\mathcal{B}$ define $(\mathcal{L}_{\mathcal{B}}, w_{\mathcal{B}}) := \boxplus_{\mathbf{b} \in \mathcal{B}}(\mathcal{L}_{\mathbf{b}}, w_{\mathbf{b}})$.

We do not need to specify the order of $\boxplus$ arithmetic in Definition 4.4 Part (2) by Lemma 3.18, which also affirms that $(\mathcal{L}_{\mathcal{B}}, w_{\mathcal{B}})$ is a weighted pseudo-LIP.

We now relate these definitions to those of Section 2.2. There, we defined a bid collection as *valid* if the associated indirect utility function is convex, and saw (Fact 2.4) that this is equivalent to, for every $\mathbf{p} \in \mathbb{R}^n$ and every $i, j \in [n]_0$ with $i \neq j$, the set $\mathcal{B}'$ of bids $\mathbf{b} \in \mathcal{B}$ such that $i, j \in \arg\max_{i \in [n]_0}(\ell(\mathbf{b})_i - p_i)$ satisfying $\sum_{\mathbf{b} \in \mathcal{B}'} m(\mathbf{b}) \geq 0$ (where we write $\ell(\mathbf{b})_0 = 0$ and $p_0 = 0$ for convenience to include the case $\mathbf{0} \in D_{\mathbf{b}}(\mathbf{p})$). But $i, j \in \arg\max_{i \in [n]_0}(\ell(\mathbf{b})_i - p_i)$ if and only if $m(\mathbf{b})\Delta_{ij} \subseteq D_{\mathbf{b}}(\mathbf{p})$, and by Lemma 4.2 this holds if and only if $\mathbf{p} \in F^i_{\mathbf{b}}$ (when $j = 0$) or $\mathbf{p} \in F^{ij}_{\mathbf{b}}$ (when $i, j \neq 0$). So, recalling that any facet which would have been weighted 0 is excluded by definition from $(\mathcal{L}_{\mathcal{B}}, w_{\mathcal{B}})$, and referring to Proposition 3.16 and Fact 2.4 Part (3), this condition is identical to:

Corollary 4.5. *Bid collection* $\mathcal{B}$ *is valid if and only if* $w_{\mathcal{B}}(F) > 0$ *on all facets* F *of* $\mathcal{L}_{\mathcal{B}}$, *which holds if and only if* $(\mathcal{L}_{\mathcal{B}}, w_{\mathcal{B}})$ *is a weighted LIP, in which case it is the LIP of the strong substitutes valuation* v *such that* $D_v(\mathbf{p}) = D_{\mathcal{B}}(\mathbf{p})$ *for all* $\mathbf{p} \in \mathbb{R}^n$.

Regarding combining bid collections, the following are clear from the definitions:

Lemma 4.6. *Let $\mathcal{B}^1$ and $\mathcal{B}^2$ be bid collections.*

(1) $(\mathcal{L}_{\mathcal{B}^1}, w_{\mathcal{B}^1}) \boxplus (\mathcal{L}_{\mathcal{B}^2}, w_{\mathcal{B}^2}) = (\mathcal{L}_{\mathcal{B}^1\cup\mathcal{B}^2}, w_{\mathcal{B}^1\cup\mathcal{B}^2})$.

(2) $(\mathcal{L}_{\mathcal{B}^1}, w_{\mathcal{B}^1}) \boxminus (\mathcal{L}_{\mathcal{B}^2}, w_{\mathcal{B}^2}) = (\mathcal{L}_{\mathcal{B}^1\cup\mathcal{B}^3}, w_{\mathcal{B}^1\cup\mathcal{B}^3})$ *where* $\mathcal{B}^3 = \{(\mathbf{r}, m) \mid (\mathbf{r}, -m) \in \mathcal{B}^2\}$.

Different bid collections define the same pseudo-LIP only in the following natural way:

Lemma 4.7. *If $\mathcal{B}^1$ and $\mathcal{B}^2$ are bid collections then $(\mathcal{L}_{\mathcal{B}^1}, w_{\mathcal{B}^1}) = (\mathcal{L}_{\mathcal{B}^2}, w_{\mathcal{B}^2})$ if and only if the sum of multiplicities of bids at any location are the same in $\mathcal{B}^1$ and in $\mathcal{B}^2$.*

Proof. For bids $\mathbf{b}^1, \mathbf{b}^2$, if $\ell(\mathbf{b}^1) = \ell(\mathbf{b}^2)$ then, writing $\mathbf{b} = (\ell(\mathbf{b}^1), m(\mathbf{b}^1) + m(\mathbf{b}^2))$, we have $\mathcal{L}_{\mathbf{b}^1} \boxplus \mathcal{L}_{\mathbf{b}^2} = \mathcal{L}_{\mathbf{b}}$. By extension over the whole bid collections, if the sum of multiplicities of bids at any location are the same in $\mathcal{B}^1$ as in $\mathcal{B}^2$ then both $(\mathcal{L}_{\mathcal{B}^1}, w_{\mathcal{B}^1})$ and $(\mathcal{L}_{\mathcal{B}^1}, w_{\mathcal{B}^1})$ are equal to the weighted LIP of a bid collection with only one bid each location, whose multiplicity is this sum of multiplicities. So $(\mathcal{L}_{\mathcal{B}^1}, w_{\mathcal{B}^1}) = (\mathcal{L}_{\mathcal{B}^2}, w_{\mathcal{B}^2})$ in this case.

Conversely, suppose that $(\mathcal{L}_{\mathcal{B}^1}, w_{\mathcal{B}^1}) = (\mathcal{L}_{\mathcal{B}^2}, w_{\mathcal{B}^2})$. Then $(\mathcal{L}_{\mathcal{B}^1}, w_{\mathcal{B}^1}) \boxminus (\mathcal{L}_{\mathcal{B}^2}, w_{\mathcal{B}^2}) = (\emptyset, 0)$, by Lemma 3.18, and so, by Lemma 4.6, we know that $\mathcal{L}_{\mathcal{B}^3} = \emptyset$ where $\mathcal{B}^3 = \mathcal{B}^1 \cup \{(\mathbf{r}, m) \mid (\mathbf{r}, -m) \in \mathcal{B}^2\}$. Suppose, for a contradiction, that $\mathbf{r}$ is minimal such that $w = \sum_{\mathbf{b}\in\mathcal{B}^3, \ell(\mathbf{b})=\mathbf{r}} m(\mathbf{b}) \neq 0$. Then, for any $i \in [n]$, there is an i-hod F of $\mathcal{L}_{\mathcal{B}^3}$ with $\bigwedge_i F = \mathbf{r}$ and with $w_{\mathcal{B}^3}(F) = w \neq 0$. But this is a contradiction as $\mathcal{L}_{\mathcal{B}^3} = \emptyset$. Thus $\sum_{\mathbf{b}\in\mathcal{B}^3, \ell(\mathbf{b})=\mathbf{r}} m(\mathbf{b}) = 0$ for all $\mathbf{r} \in \mathbb{R}^n$, whence by definition of $\mathcal{B}^3$ we see that the sum of multiplicities of bids at any location are the same in $\mathcal{B}^1$ and $\mathcal{B}^2$. □

Lemma 4.7 shows that there is essentially only one bid collection giving rise to any pseudo-LIP, or, by Corollary 4.5, to any LIP; it therefore shows that Theorem 2.9 follows as an immediate Corollary from Theorems 2.6–2.8. As we discuss in Section 2.2, bid collections are unique under either of two possible conventions; for the purposes of this paper we are agnostic between these conventions and so bid collections will not be truly unique.

4.3 The Bounding Box (High prices)

Recall from Fact 2.5 that if $\mathcal{B}$ is a valid bid collection, then the domain of the corresponding valuation $v_{\mathcal{B}}$ is a simplex. We can now see more clearly the source of the problem. Observe from Lemma 4.2 that the i-hod and ij-fin of a bid are respectively bounded below with respect to $\leq_i$ and $\leq_{ij}$. This same properties will therefore hold for all i-hods and ij-fin of any pseudo-LIP $\mathcal{L}_{\mathcal{B}}$ of any bid collection $\mathcal{B}$, including when $\mathcal{L}_{\mathcal{B}}$ is a LIP. By Corollary 3.11, if a strong substitutes valuation v has simplex domain, then it also satisfies this property, but a general strong substitutes LIP need not.

We handle these cases by using bids in a "bounding box", which we think of as being "sufficiently high" prices that all the interest of the LIP is contained inside. The valuation will be correctly presented at any price inside this box. Section 4.4 shows how we can re-interpret bids on the boundary of this box, so that the presentation is precise for all prices.

Recall that for high prices $H \gg 0$, we introduced in Section 2.2 the notation $\mathbf{H} := [-H, H]^n$, with $\mathbf{H}^\circ := (-H, H)^n$, and $\partial\mathbf{H} := \mathbf{H} \setminus \mathbf{H}^\circ$. We now define additionally:

Definition 4.8. For *high prices* $H \gg 0$, write:

(1) $\overline{\partial}^i\mathbf{H} := \mathbf{H} \cap \{\mathbf{p} \in \mathbb{R}^n \,|\, p_i = H\}$
(2) $\underline{\partial}^i\mathbf{H} := \mathbf{H} \cap \{\mathbf{p} \in \mathbb{R}^n \,|\, p_i = -H\}$
(3) $\mathcal{L}_\mathbf{H} := \bigcup_{i\in[n]} (\{\mathbf{p} \in \mathbb{R}^n \,|\, p_i = -H\} \cup \{\mathbf{p} \in \mathbb{R}^n \,|\, p_i = H\})$.

Note that $\mathcal{L}_\mathbf{H}$ is the union of affine spans of faces of $\overline{\partial}^i\mathbf{H}$ and $\underline{\partial}^i\mathbf{H}$ for all i and is, by Fact 3.5, itself the LIP of a valuation if we assign, for example, weight 1 to every facet (whereas $\partial\mathbf{H}$ is not the LIP of any valuation).

We will assume that H satisfies the following relative to the LIP $\mathcal{L}_v$ in question:

Assumption 4.9. $\mathbf{H}^\circ \cap C \neq \emptyset$ for every cell C of $\mathcal{L}_v$.

Such H always exists because there are only finitely many cells of any LIP $\mathcal{L}_v$. This property allows us to uniquely identify our LIP from its intersection with $\mathbf{H}^\circ$:

Proposition 4.10. *Suppose $H \gg 0$ satisfies Assumption 4.9 for both $\mathcal{L}_{v^1}$ and $\mathcal{L}_{v^2}$. If $\mathcal{L}_{v^1} \cap \mathbf{H}^\circ = \mathcal{L}_{v^2} \cap \mathbf{H}^\circ$ then $\mathcal{L}_{v^1} = \mathcal{L}_{v^2}$.*

And, within $\mathbf{H}^\circ$, we can identify our LIP with one with a simplex domain:

Proposition 4.11. *Let $v : A \to \mathbb{R}$ be a strong substitutes valuation, let H satisfy Assumption 4.9 for $\mathcal{L}_v$, and let $D \in \mathbb{Z}_{>0}$ be minimal such that $A \subseteq D\Delta_{[n_0]}$. Then there exists a strong substitutes valuation $\widehat{v} : D\Delta_{[n_0]} \to \mathbb{R}$ such that $D_v(\mathbf{p}) = D_{\widehat{v}}(\mathbf{p})$ for all $\mathbf{p} \in \mathbf{H}^\circ$, and such that $\bigwedge_i F^i, \bigwedge_{ij} F^{ij} \in \mathbf{H}$ for all i-hods F^i and ij-fin F^{ij} of $\mathcal{L}_{\widehat{v}}$.*

4.4 Re-Interpretation of Boundary Bids

We can use Proposition 4.11 to give representations which hold in an arbitrarily large bounded subset $\mathbf{H}$ of $\mathbb{R}^n$. But some of the bids in such a representation will fall on the boundary of $\mathbf{H}$. As outlined in Section 2.2, we can re-interpret such bids by using them to define pseudo-LIPs $\mathcal{L}_\mathbf{b}^\mathbf{H}$ that satisfy Assumption 4.9, and whose intersection with $\mathbf{H}^\circ$ is identical to that of $\mathcal{L}_\mathbf{b}$. The representation of these simpler LIPs as bids on the boundary of a box is useful because it allows us to concisely identify the location of these simpler LIPs (which have no 0-cell) in $\mathbb{R}^n$.

We repeat from Section 2.2 the definition of $S(\ell(\mathbf{b}), \mathbf{H})$:

$$S(\ell(\mathbf{b}), \mathbf{H}) = \begin{cases} \{i \in [n] \mid \ell(\mathbf{b}) \notin \underline{\partial}^i\mathbf{H}\} \cup \{0\} & \text{if } \ell(\mathbf{b}) \notin \bigcup_{j\in[n]} \overline{\partial}^j\mathbf{H} \\ \{i \in [n] \mid \ell(\mathbf{b}) \notin \underline{\partial}^i\mathbf{H}\} & \text{if } \ell(\mathbf{b}) \in \bigcup_{j\in[n]} \overline{\partial}^j\mathbf{H} \end{cases} \tag{5}$$

We write $v_\mathbf{b}^\mathbf{H} := v_\mathbf{b}|_{\Delta_{S(\ell(\mathbf{b}),\mathbf{H})}}$, and write $(\mathcal{L}_\mathbf{b}^\mathbf{H}, w_\mathbf{b}^\mathbf{H}) := (\mathcal{L}_{v_\mathbf{b}^\mathbf{H}}, w_{v_\mathbf{b}^\mathbf{H}})$ when $m(\mathbf{b}) > 0$; if $m(\mathbf{b}) < 0$ we set $(\mathcal{L}_\mathbf{b}^\mathbf{H}, w_\mathbf{b}^\mathbf{H}) = (\mathcal{L}_{|\mathbf{b}|}^\mathbf{H}, -w_{|\mathbf{b}|})$ as usual.

Lemma 4.12. *Suppose bid $\mathbf{b}$ satisfies $\ell(\mathbf{b}) \in \mathbf{H}$. Then:*

(1) $(\mathcal{L}_\mathbf{b} \cap \mathbf{H}^\circ, w_\mathbf{b}) = (\mathcal{L}_\mathbf{b}^\mathbf{H} \cap \mathbf{H}^\circ, w_\mathbf{b}^\mathbf{H})$;
(2) H satisfies Assumption 4.9 for $\mathcal{L}_\mathbf{b}^\mathbf{H}$;
(3) $D_\mathbf{b}^\mathbf{H}(\mathbf{p}) = D_\mathbf{b}(\mathbf{p})$ for all $\mathbf{p} \in \mathbf{H}^\circ$;
(4) If additionally $\ell(\mathbf{b}) \in \mathbf{H}^\circ$ then $D_\mathbf{b}^\mathbf{H}(\mathbf{p}) = D_\mathbf{b}(\mathbf{p})$ for all $\mathbf{p} \in \mathbb{R}^n$.

This alternative interpretation of the boundary bids enables us to see that if $\mathcal{L}_v$ has been presented in $\mathbf{H}^\circ$ by bids in $\mathbf{H}$, then we can infer the shape of $\mathcal{L}_v$ everywhere:

Corollary 4.13. *If v is a strong substitutes valuation and H satisfies Assumption 4.9, and if $\mathcal{B}$ satisfies $\ell(\mathbf{b}) \in \mathbf{H}$ for all $\mathbf{b} \in \mathcal{B}$ and $(\mathcal{L}_v \cap \mathbf{H}^\circ, w_v) = (\mathcal{L}_\mathcal{B} \cap \mathbf{H}^\circ, w_\mathcal{B})$, then*

$$(\mathcal{L}_v, w_v) = \boxplus_{\mathbf{b}\in\mathcal{B}}(\mathcal{L}^{\mathbf{H}}_{\mathbf{b}}, w^{\mathbf{H}}_{\mathbf{b}}) \tag{6}$$

Proof. By assumption $(\mathcal{L}_v \cap \mathbf{H}^\circ, w_v) = (\mathcal{L}_\mathcal{B} \cap \mathbf{H}^\circ, w_\mathcal{B}) = \boxplus_{\mathbf{b}\in\mathcal{B}}(\mathcal{L}_\mathbf{b} \cap \mathbf{H}^\circ, w_\mathbf{b})$. By Lemma 4.12 we know $(\mathcal{L}_\mathbf{b} \cap \mathbf{H}^\circ, w_\mathbf{b}) = (\mathcal{L}^{\mathbf{H}}_{\mathbf{b}} \cap \mathbf{H}^\circ, w^{\mathbf{H}}_{\mathbf{b}})$ for all $\mathbf{b} \in \mathcal{B}$, and that $\mathcal{L}^{\mathbf{H}}_{\mathbf{b}}$ satisfies Assumption 4.9 for all $\mathbf{b} \in \mathcal{B}$. So the result follows from Proposition 4.10 if we can also see that $\boxplus_{\mathbf{b}\in\mathcal{B}}(\mathcal{L}^{\mathbf{H}}_{\mathbf{b}}, w^{\mathbf{H}}_{\mathbf{b}})$ satisfies Assumption 4.9. But we can observe from Lemma 4.2 that if $\mathbf{b}^1, \mathbf{b}^2 \in \mathbf{H}^\circ$ then every cell in $\mathcal{L}_{\mathbf{b}^1} \cap \mathcal{L}_{\mathbf{b}^2}$ has non-zero intersection with $\mathbf{H}^\circ$, and by definition of the LIPs $\mathcal{L}^{\mathbf{H}}_{\mathbf{b}}$, the same follows for these. Equation (6) follows. □

5 Proof of Main Theorems

5.1 The Structure of the Proof

We focus on LIPs with simplex domain. We construct a set of positive and negative bids generating a strong substitutes LIP $\mathcal{L}_v$ by iteratively covering a series of LIPs, as follows.

(1) Set $\mathcal{L}_{v^0} := \mathcal{L}_v$.
(2) Find $\mathcal{B}^s$ which covers $\mathcal{L}_{v^s}$. Set $(\mathcal{L}_{v^{s+1}}, w_{v^{s+1}}) := (\mathcal{L}_{\mathcal{B}^s}, w_{\mathcal{B}^s}) \boxminus (\mathcal{L}_{v^s}, w_{v^s})$
(3) Let $\mathcal{B} = \bigcup_{t=0}^{\infty} \mathcal{B}^{2t} \cup \bigcup_{t=0}^{\infty}\{(\mathbf{r}, -m) \mid (\mathbf{r}, m) \in \mathcal{B}^{2t+1}\}$.

That is, we first find positive bids $\mathcal{B}^0$ covering $\mathcal{L}_v$; these shall be positive in our final bid collection. But the LIP $\mathcal{L}_{\mathcal{B}^0}$ generated by $\mathcal{B}^0$ is in general excessive to generate $\mathcal{L}_v$. We seek to subtract the excess parts; to identify what must be subtracted, we work the other way around. That is, we subtract from $\mathcal{L}_{\mathcal{B}^0}$ the LIP $\mathcal{L}_v$ which we *did* wish to cover, and identify the remainder, which we label $\mathcal{L}_{v^1}$. We seek a cover $\mathcal{B}^1$ of this. Therefore, although $\mathcal{L}_{v^1}$ is itself a (positive-weighted) LIP, and the cover $\mathcal{B}^1$ consists of positive bids, these correspond to bids that will be *negative* in our final bid collection. We then repeat the process; the natural "double negative" property means that we switch between identifying bids which will be positive in the final outcome, and those which will be negative. The set $\mathcal{B}$ presented above will in general have many redundancies, but we can eliminate these easily, by taking the signed sum of the multiplicities of bids at any location.

However, for this construction to make sense as a finite collection of bids, we must show that $\mathcal{B}^s \neq \emptyset$ for only finitely many values of s. Then:

Proposition 5.1. *Suppose $(\mathcal{B}^s)_{s=0}^{\infty}$ and $(\mathcal{L}_{v^s}, w_{v^s})_{s=0}^{\infty}$ are lists respectively of finite collections of positive bids, and weighted LIPs, such that, for all $s \geq 0$, we have that $(\mathcal{L}_{v^s}, w_{v^s}) \preceq (\mathcal{B}^s, w_{\mathcal{B}^s})$, and that $(\mathcal{L}_{v^{s+1}}, w_{v^{s+1}}) = (\mathcal{L}_{\mathcal{B}^s}, w_{\mathcal{B}^s}) \boxminus (\mathcal{L}_{v^s}, w_{v^s})$. Suppose moreover that $\mathcal{B}^s \neq \emptyset$ for only finitely many values of s.*

Then $(\mathcal{L}_\mathcal{B}, w_\mathcal{B}) = (\mathcal{L}_{v^0}, w_{v^0})$, *where*

$$\mathcal{B} = \bigcup_{t=0}^{\infty} \mathcal{B}^{2t} \cup \bigcup_{t=0}^{\infty} \{(\mathbf{r}, -m) \mid (\mathbf{r}, m) \in \mathcal{B}^{2t+1}\}. \tag{7}$$

is a valid bid collection.

Proof. We know that $\mathcal{B}^s \neq \emptyset$ for only finitely many values of s; it is convenient to write this as $\mathcal{B}^s = \emptyset$ for all $s \geq 2S+1$ for some $S \in \mathbb{Z}_{>0}$. As $\mathcal{B}$ is then formed from finitely many (finite) collections of bids, it is also a finite bid collection. Since $(\mathcal{L}_{v^{2S+1}}, w_{v^{2S+1}}) \preceq (\mathcal{B}^{2S+1}, w_{\mathcal{B}^{2S+1}}) = (\emptyset, 0)$ it follows that $\mathcal{L}_{v^{2S+1}} = \emptyset$. Repeatedly applying the definition of $\mathcal{L}_{v^s}$ and Lemma 3.18, we see:

$$\begin{aligned}
(\emptyset, 0) = (\mathcal{L}_{v^{2S+1}}, w_{v^{2S+1}}) &= (\mathcal{L}_{\mathcal{B}^{2S}}, w_{\mathcal{B}^{2S}}) \boxminus (\mathcal{L}_{v^{2S}}, w_{v^{2S}}) \\
&= (\mathcal{L}_{\mathcal{B}^{2S}}, w_{\mathcal{B}^{2S}}) \boxminus ((\mathcal{L}_{\mathcal{B}^{2S-1}}, w_{\mathcal{B}^{2S-1}}) \boxminus (\mathcal{L}_{v^{2S-1}}, w_{v^{2S-1}})) \\
&= ((\mathcal{L}_{\mathcal{B}^{2S}}, w_{\mathcal{B}^{2S}}) \boxminus (\mathcal{L}_{\mathcal{B}^{2S-1}}, w_{\mathcal{B}^{2S-1}})) \boxplus (\mathcal{L}_{v^{2S-1}}, w_{v^{2S-1}}) \\
&= \cdots \\
&= \boxplus_{t=0}^{S} (\mathcal{L}_{\mathcal{B}^{2S-2t}}, w_{\mathcal{B}^{2S-2t}}) \boxminus (\boxplus_{t=0}^{S-1} (\mathcal{L}_{\mathcal{B}^{2S-2t-1}}, w_{\mathcal{B}^{2S-2t-1}})) \boxminus (\mathcal{L}_{v^0}, w_{v^0}).
\end{aligned}$$

And so, taking $\boxplus(\mathcal{L}_{v^0}, w_{v^0})$ on both sides, and applying the definition of $\mathcal{B}$, we have:

$$(\mathcal{L}_{v^0}, w_{v^0}) = (\mathcal{L}_\mathcal{B}, w_\mathcal{B})$$

as required. This demonstrates that $w_\mathcal{B}(F) > 0$ for all facets F of $\mathcal{L}_\mathcal{B}$, and so validity of $\mathcal{B}$ follows by Corollary 4.5. □

The set $\mathcal{B}$ presented in Proposition 5.1 (Equation (7)) may contain may redundancies; by Lemma 4.7 it also follows that $(\mathcal{L}_{v^0} \cap \mathbf{H}^\circ, w_{v^0}) = (\mathcal{L}_{\mathcal{B}'} \cap \mathbf{H}^\circ, w_{\mathcal{B}'})$, where:

$$\mathcal{B}' = \left\{ (\mathbf{r}, m) \mid m = \sum_{\mathbf{b} \in \mathcal{B}, \ell(\mathbf{b}) = \mathbf{r}} m(\mathbf{b}),\, m \neq 0 \right\}.$$

It remains to show that the objects described by Proposition 5.1 do indeed exist.

Proposition 5.2. *For any strong substitutes valuation v with simplex domain, there exist lists $(\mathcal{B}^s)_{s=0}^{\infty}$ and $(\mathcal{L}_{v^s}, w_{v^s})_{s=0}^{\infty}$ of respectively finite collections of positive bids, and weighted pseudo-LIPs, such that $(\mathcal{L}_{v^0}, w_{v^0}) = (\mathcal{L}_v, w_v)$, such that for all $s \geq 0$, we have that $(\mathcal{L}_{v^s}, w_{v^s}) \preceq (\mathcal{B}^s, w_{\mathcal{B}^s})$ and $(\mathcal{L}_{v^{s+1}}, w_{v^{s+1}}) = (\mathcal{L}_{\mathcal{B}^s}, w_{\mathcal{B}^s}) \boxminus (\mathcal{L}_{v^s}, w_{v^s})$, and such that $\mathcal{B}^s \neq \emptyset$ for only finitely many values of s. Moreover, if every i-hod F^i and ij-fin F^{ij} of $\mathcal{L}_{v^0}$ satisfy $\bigwedge_i F^i, \bigwedge_{ij} F^{ij} \in \mathbf{H}$ then $\ell(\mathbf{b}) \in \mathbf{H}$ for all $\mathbf{b} \in \bigcup_{s \geq 0} \mathcal{B}^s$.*

5.2 Proofs of the Main Theorems, Contingent on Proposition 5.2

Before providing the bid collections $\mathcal{B}^s$ described in Proposition 5.2, we first show that this is indeed the final step in proving our main results.

Proof of Theorem 2.6. By Proposition 5.1 and Proposition 5.2, there exists a valid collection $\mathcal{B}$ of bids such that $(\mathcal{L}_{\mathcal{B}}, w_{\mathcal{B}}) = (\mathcal{L}_v, w_v)$. As $\mathbf{0} \in D\Delta_{[n_0]}$, it follows that $D_v(\mathbf{p}) = \{\mathbf{0}\}$ if all coordinates of $\mathbf{p}$ are sufficiently large. Also $D_{\mathcal{B}}(\mathbf{p}) = \{\mathbf{0}\}$ if $p_i > (\ell(\mathbf{b}))_i$ for all $i \in [n]$ and all $\mathbf{b} \in \mathcal{B}$. Now Fact 3.5 confirms that $v_{\mathcal{B}} = v$. □

Proof of Theorem 2.7. By Proposition 4.11, there exists a strong substitutes valuation with simplex domain such that $D_v(\mathbf{p}) = D_{\widehat{v}}(\mathbf{p})$ for all $\mathbf{p} \in \mathbf{H}^\circ$, and such that $\bigwedge_i F^i, \bigwedge_{ij} F^{ij} \in \mathbf{H}$ for all i-hods F^i and ij-fin F^{ij} of $\mathcal{L}_{\widehat{v}}$. It follows by Theorem 2.6 that there exists a valid bid collection $\mathcal{B}$ such that $v_{\mathcal{B}} = \widehat{v}$. Thus $D_v(\mathbf{p}) = D_{\mathcal{B}}(\mathbf{p})$ for all $\mathbf{p} \in \mathbf{H}^\circ$. Additionally, Proposition 5.2 confirms that $\ell(\mathbf{b}) \in \mathbf{H}$ for all $\mathbf{b} \in \mathcal{B}$. □

Proof of Theorem 2.8. This now follows from Theorem 2.7 and Corollary 4.13. □

Proof of Theorem 2.9. This now follows from Theorem 2.7 and Lemma 4.7. □

5.3 The Proof of Proposition 5.2

It remains to identify suitable sets $\mathcal{B}^s$, and show that only finitely many of these sets are non-empty. To facilitate this, we identify a finite set of points, at which all our bids will lie. First, if F is a polyhedral set in $\mathbb{R}^n$, write $\langle F \rangle$ for the affine span of F. Now define:

Definition 5.3. Suppose v is a strong substitutes valuation with simplex domain.

(1) Write $\langle \mathcal{L}_v \rangle$ for the union of sets $\langle F \rangle$ where F is hod of $\mathcal{L}_v$.
(2) Define the *grid points* of $\mathcal{L}_v$ to be the 0-cells of $\langle \mathcal{L}_v \rangle$.

Observe that, because any LIP $\mathcal{L}_v$ has only finitely many facets, $\langle \mathcal{L}_v \rangle$ is indeed also a finite rational polyhedral complex and in particular there are only finitely many grid points. If we endow every facet of $\langle \mathcal{L}_v \rangle$ with weight 1, then this has the structure of a balanced weighted rational polyhedral complex, and is thus a LIP. The following is clear:

Lemma 5.4. *If every i-hod F^i and ij-fin F^{ij} of $\mathcal{L}_v$ satisfy $\bigwedge_i F^i, \bigwedge_{ij} F^{ij} \in \mathbf{H}$ then every grid point of $\mathcal{L}_v$ is in $\mathbf{H}$.*

Grid points of $\mathcal{L}_v$ need not be 0-cells of $\mathcal{L}_v$, and indeed need not be contained in $\mathcal{L}_v$. We will see that $\mathcal{L}_v$ can be generated using only bids located at grid points.

Observe that the facets of $\langle \mathcal{L}_v \rangle$ are all products of intervals (we might call them "hyper-rectangles"). We show that the collection of hods of $\mathcal{L}_v$ are built up out of the facets of $\langle \mathcal{L}_v \rangle$, in the following way:

Lemma 5.5. *Let $\mathcal{L}_v$ be a strong substitutes LIP with simplex domain.*

(1) If F is an i-hod of $\mathcal{L}_v$ then $F \subseteq \langle \mathcal{L}_v \rangle$
(2) If F is a facet of $\langle \mathcal{L}_v \rangle$ such that $F \cap \mathcal{L}_v$ is $(n-1)$-dimensional, then $F \subseteq \mathcal{L}_v$. Moreover, then all facets F' of $\mathcal{L}_v$ such that $F' \cap F$ is $(n-1)$-dimensional have the same weight.

We can now go on to identify the bids for each stage of our construction:

Definition 5.6. Given a strong substitutes LIP $\mathcal{L}_v$ with simplex domain, we set $\mathcal{L}_{v^0} = \mathcal{L}_v$ and for $s \geq 0$ inductively define:

(1) $\mathcal{B}^s$ to be the bids $(\mathbf{r}, m)$ where:
  (i) $\mathbf{r}$ is a grid point for $\mathcal{L}_v$ and $\mathbf{r} = \bigwedge_i F$ where F an i-hod of $\langle \mathcal{L}_v \rangle$ such that $F \subseteq \mathcal{L}_{v^s}$;
  (ii) m is maximal such $m = w_{v^s}(F')$ for some facet F' of $\mathcal{L}_{v^s}$ such that $F' \cap F$ is n-dimensional, for some i-hod F of $\langle \mathcal{L}_v \rangle$ as in (1)(i);

(2) $(\mathcal{L}_{v^{s+1}}, w_{v^{s+1}}) := (\mathcal{L}_{\mathcal{B}^s}, w_{\mathcal{B}^s}) \boxminus (\mathcal{L}_{v^s}, w_{v^s})$.

It immediately follows from this definition that:

Corollary 5.7. *If* $\mathbf{b} \in \mathcal{B}^s$ *then* $\mathcal{L}_{\mathcal{B}^s}$ *contains a hod* F *with* $\ell(\mathbf{b}) \in F$*.*

We relegate the proof of the following important technical result to the Appendix.

Proposition 5.8. $(\mathcal{L}_{v^s}, w_{v^s})$ *is a weighted strong substitutes LIP with simplex domain and* $(\mathcal{L}_{v^s}, w_{v^s}) \preceq (\mathcal{L}_{\mathcal{B}^s}, w_{\mathcal{B}^s})$*.*

Thus, at each stage, we cover the LIP with a set of bids, whose locations are always drawn from the same finite set.

It remains to show that $\mathcal{B}^s \neq \emptyset$ for only finitely many values of s. We do so by showing that if the locations of bids in $\mathcal{B}^s$ which are minimal with respect to the Euclidean ordering, strictly increases (with respect to the Euclidean ordering) at each stage.

First, a minimal bid in one bid collection cannot be above a minimal bid in the next, with respect to the Euclidean ordering, because:

Lemma 5.9. *For any* $s \geq 0$*, if* $\mathbf{b} \in \mathcal{B}^{s+1}$ *then there exists* $\mathbf{b}' \in \mathcal{B}^s$ *with* $\ell(\mathbf{b}') \leq \ell(\mathbf{b})$*.*

Proof. If $\mathbf{b} \in \mathcal{B}^{s+1}$ then $\ell(\mathbf{b})$ is a grid point for $\mathcal{L}_v$, and additionally by Corollary 5.7 $\ell(\mathbf{b}) \in F$ where F is a hod of $\mathcal{L}_{v^{s+1}}$. But $(\mathcal{L}_{v^{s+1}}, w_{v^{s+1}}) = (\mathcal{L}_{\mathcal{B}^s}, w_{\mathcal{B}^s}) \boxminus (\mathcal{L}_{v^s}, w_{v^s})$ and $\mathcal{L}_{v^s}, \mathcal{L}_{v^{s+1}}$ are both (positive) weighted LIPs. So there must exist a facet F' of $\mathcal{L}_{\mathcal{B}^s}$ such that $F \cap F'$ has dimension $n-1$, and such that $\ell(\mathbf{b}) \in F'$. By definition of $\mathcal{L}_{\mathcal{B}^s}$ it follows that there exists $\mathbf{b}' \in \mathcal{B}^s$ with $\ell(\mathbf{b}') \leq_i \ell(\mathbf{b})$, and hence $\ell(\mathbf{b}') \leq \ell(\mathbf{b})$. □

Next, to show that the minimal bid in one step must be strictly below the minimal bid in the next, we will need to know (proof in Appendix):

Lemma 5.10. *For any* $s \geq 0$*, if* $\ell(\mathbf{b})$ *is minimal with respect to the Euclidean ordering subject to* $\mathbf{b} \in \mathcal{B}^s$ *then* $(\mathcal{L}_{\mathcal{B}^{s+1}}, w_{\mathcal{B}^{s+1}}) = (\mathcal{L}_{\mathcal{B}^s}, w_{\mathcal{B}^s}) \boxminus (\mathcal{L}_{v^s}, w_{v^s})$ *has no hods containing* $\ell(\mathbf{b})$*.*

We can now prove:

Proposition 5.11. *For any* $s \geq 0$*, if* $\ell(\mathbf{b})$ *is minimal with respect to the Euclidean ordering subject to* $\mathbf{b} \in \mathcal{B}^s$ *then for all* $t \geq 1$ *there does not exist* $\mathbf{b}' \in \mathcal{B}^{s+1}$ *with* $\ell(\mathbf{b}') \leq \ell(\mathbf{b})$*.*

Proof. By Lemma 5.10 we know that $(\mathcal{L}_{\mathcal{B}^{s+1}}, w_{\mathcal{B}^{s+1}}) = (\mathcal{L}_{\mathcal{B}^s}, w_{\mathcal{B}^s}) \boxminus (\mathcal{L}_{v^s}, w_{v^s})$ does not contain a hod containing $\ell(\mathbf{b})$. Applying Corollary 5.7, then, $\mathcal{B}^{s+1}$ does not contain a bid $\mathbf{b}'$ with $\ell(\mathbf{b}') = \ell(\mathbf{b})$. Now by Lemma 5.9, since $\ell(\mathbf{b})$ is minimal in $\mathcal{B}^s$ with respect to $\leq$, there cannot be $\mathbf{b}' \in \mathcal{B}^{s+1}$ with $\ell(\mathbf{b}') \leq \ell(\mathbf{b})$. □

Corollary 5.12. *$\mathcal{B}^s \neq \emptyset$ for finitely many values of s.*

Proof. We define a series of possible locations for bids at each stage. Let G_0 be the set of all grid points. For $s \in \mathbb{Z}_{\geq 0}$, write

$$G_{s+1} := \{\mathbf{r} \in G_0 \mid \exists \mathbf{b} \in \mathcal{B}^s \text{ with } \ell(\mathbf{b}) \leq \mathbf{r} \text{ and } \ell(\mathbf{b}) \neq \mathbf{r}\}.$$

We claim that $\ell(\mathbf{b}) \in G_s$ for all $\mathbf{b} \in \mathcal{B}^s$. The case $s = 0$ is clear by definition of $\mathcal{B}^0$. Consider $\mathbf{b} \in \mathcal{B}^{s+1}$, and let $\mathbf{b}' \in \mathcal{B}^s$ be minimal with respect to $\leq$ such that $\ell(\mathbf{b}') \leq \ell(\mathbf{b})$. By Lemma 5.9 such $\mathbf{b}'$ exists, and by Proposition 5.11 we know $\ell(\mathbf{b}') \neq \ell(\mathbf{b})$. So indeed $\ell(\mathbf{b}) \in G_{s+1}$.

We now show that $G_{s+1} \subseteq G_s$ for all $s \in \mathbb{Z}_{\geq 0}$. Clearly $G_1 \subseteq G_0$. To show that $G_{s+2} \subseteq G_{s+1}$ for all $s \in \mathbb{Z}_{\geq 0}$, let $\mathbf{r} \in G_{s+2}$. Then there exists $\mathbf{b} \in \mathcal{B}^{s+1}$ with $\ell(\mathbf{b}) \leq \mathbf{r}$ and $\ell(\mathbf{b}) \neq \mathbf{r}$, whence by Lemma 5.9 there exists $\mathbf{b}' \in \mathcal{B}^s$ with $\ell(\mathbf{b}') \leq \ell(\mathbf{b})$. Thus $\ell(\mathbf{b}') \leq \mathbf{r}$, and $\ell(\mathbf{b}') = \mathbf{r}$ would imply the contradiction $\ell(\mathbf{b}) = \mathbf{r}$; so $\ell(\mathbf{b}') \neq \mathbf{r}$ and hence $\mathbf{r} \in G_{s+1}$.

But moreover, if $G_{s+1} \neq \emptyset$ then $G_{s+1} \neq G_s$. To see this, observe that $G_{s+1} \neq \emptyset$ only if $\mathcal{B}^s \neq \emptyset$; but if $\mathbf{b} \in \mathcal{B}^s$ and $\ell(\mathbf{b})$ is minimal with respect to $\leq$, then $\ell(\mathbf{b}) \in G_s$ and $\ell(\mathbf{b}) \notin G_{s+1}$.

Thus we have a chain of sets $G_0 \supseteq G_1 \supseteq \cdots$, with the inclusions being strict while the sets are non-empty. The first set in this chain, $|G_0|$, is finite. It follows that the sets G_s are nonempty for finitely many values of s. But if $G_s = \emptyset$ then $\mathcal{B}^s = \emptyset$, so this completes the proof. □

Proof of Proposition 5.2. Given v with simplex domain, Definition 5.6 provides lists $(\mathcal{B}^s)_{s=0}^{\infty}$ and $(\mathcal{L}_{v^s}, w_{v^s})_{s=0}^{\infty}$ of respectively finite collections of positive bids, and weighted pseudo-LIPs, such that $(\mathcal{L}_{v^0}, w_{v^0}) = (\mathcal{L}_v, w_v)$ and such that, for all $s \geq 0$, we have that $(\mathcal{L}_{v^{s+1}}, w_{v^{s+1}}) = (\mathcal{L}_{\mathcal{B}^s}, w_{\mathcal{B}^s}) \boxminus (\mathcal{L}_{v^s}, w_{v^s})$. Proposition 5.8 then shows that $(\mathcal{L}_{v^s}, w_{v^s})$ is indeed a weighted LIP and $(\mathcal{L}_{v^s}, w_{v^s}) \preceq (\mathcal{L}_{\mathcal{B}^s}, w_{\mathcal{B}^s}))$. We know $\mathcal{B}^s \neq \emptyset$ for only finitely many values of s, by Corollary 5.12. Every bid in $\mathbf{b} \in \bigcup_{s \geq 0} \mathcal{B}^s$ satisfies $\ell(\mathbf{b})$ is a grid point of $\mathcal{L}_v$, and therefore, by Lemma 5.4, is in $\mathbf{H}$. This completes the proof. □

A The Geometry of Strong Substitutes LIPs

This appendix develops results which will be used in our proofs in Appendix B.

A.1 Possible $(n-2)$-cells for Strong Substitutes LIPs

Recall that if C is a polyhedral set, we write $\langle C \rangle$ for the affine span of C.

Definition A.1. For any distinct indices $i, j, k, l \in [n]$, we say that an $(n-2)$-cell C of a LIP $\mathcal{L}_v$ is:

(1) *Type 1 with indices i, j*, if $\langle C \rangle = \{\mathbf{p} \in \mathbb{R}^n \mid p_i = r_i;\ p_j = r_j\}$ for some $\mathbf{r} \in \mathbb{R}^n$;
(2) *Type 2 with indices i, j, k*, if $\langle C \rangle = \{\mathbf{p} \in \mathbb{R}^n \mid (p_i - r_i) = (p_j - r_j) = (p_k - r_k)\}$ for some $\mathbf{r} \in \mathbb{R}^n$;
(3) *Type 3 with flat index i and skew indices j, k* if $\langle C \rangle = \{\mathbf{p} \in \mathbb{R}^n \mid p_i = r_i;\ (p_j - r_j) = (p_k - r_k)\}$ for some $\mathbf{r} \in \mathbb{R}^n$;

(4) *Type 4 with index pairs i, j and k, l* if $\langle C \rangle = \{\mathbf{p} \in \mathbb{R}^n : a_i(p_i - r_i) = a_j(p_j - r_j);\ a_k(p_k - r_k) = a_l(p_l - r_l)\}$ for some $\mathbf{r} \in \mathbb{R}^n$.

Lemma A.2. *Let $\mathcal{L}_v$ be a strong substitutes LIP, let F be a facet of $\mathcal{L}_v$ and let $C \subsetneq F$ be an $(n-2)$-cell. Then C is one of Types 1,2,3 and 4 from Definition A.1. The possible form of F depends on the Type of C as follows:*

(1) If C is Type 1 with indices i, j then F is an i-hod, a j-hod or an ij-fin.
(2) If C is Type 2 with indices i, j, k then F is an ij-fin, an ik-fin or a jk-fin.
(3) If C is Type 3 with flat index i and skew indices j, k then F is an i-hod or a jk-fin.
(4) If C is Type 4 with index pairs i, j and k, l then F is a ij-fin or a kl-fin.

Proof. An $(n-2)$-cell is the intersection of (at least) two non-parallel facets F^1, F^2. As there are limited possible facet normals, we may break this down into 5 cases. Consideration of these cases together proves the Lemma.

Case 1: F^1 has normal $\mathbf{e}^i$ and F^2 has normal $\mathbf{e}^j$ where $i \neq j$. Here C is Type 1 with indices i, j.

Case 2: F^1 has normal $\mathbf{e}^i$ and F^2 has normal $\mathbf{e}^i - \mathbf{e}^j$, for $j \neq i$. The space of vectors normal to $\langle C \rangle$ is spanned by $\{\mathbf{e}^i, \mathbf{e}^j\}$, so again C is Type 1 with indices i, j.

Case 3: F^1 has normal $\mathbf{e}^i$ and F^2 has normal $\mathbf{e}^j - \mathbf{e}^k$ for distinct i, j, k. The space of vectors normal to $\langle C \rangle$ is spanned by $\{\mathbf{e}^i, \mathbf{e}^j - \mathbf{e}^k\}$, and so C is Type 3 with flat index i and skew indices j, k.

Case 4. F^1 has normal $\mathbf{e}^i - \mathbf{e}^j$ and F^2 has normal $\mathbf{e}^j - \mathbf{e}^k$ where i, j, k are distinct. The space of vectors normal to $\langle C \rangle$ is spanned by $\{\mathbf{e}^i - \mathbf{e}^j, \mathbf{e}^j - \mathbf{e}^k\}$ and also contains $\mathbf{e}^i - \mathbf{e}^k$, so C is Type 2 with indices i, j, k.

Case 5. F^1 has normal $\mathbf{e}^i - \mathbf{e}^j$ and F^2 has normal $\mathbf{e}^k - \mathbf{e}^l$ where i, j, k, l are distinct. The space of vectors normal to $\langle C \rangle$ is spanned by $\{\mathbf{e}^i - \mathbf{e}^j, \mathbf{e}^k - \mathbf{e}^l\}$, so C is Type 4 with index pairs i, j and k, l. □

Recall that every LIP is balanced, when paired with the facet weights (Definition 3.3 and Fact 3.5).

Corollary A.3. *Let $\mathcal{L}_v$ be a strong substitutes LIP, and let C be an $(n-2)$-cell such that:*

(1) C is of Type 1 or 2, but facets containing C only represents two of the three distinct hod or fin types as listed in Lemma A.2;
(2) C is of Type 3 or 4.

Then if $\mathcal{L}_v$ has a facet F with normal $\mathbf{d}$ containing C in its boundary, it follows that $\mathcal{L}_v$ also has a facet $F' \neq F$ with weight $w_v(F)$ and normal $\mathbf{d}$ containing C in its boundary.

Proof. Observe from Lemma A.2 that in all of the cases listed, there are only two possible normal vectors for facets containing such C in their boundary. These normal vectors are linearly independent. The balancing condition therefore tells us that the weight of a facet with this normal on one side of the bounding $(n-2)$-cell must be exactly equal to the weight of a facet with this normal on the other side of the bounding $(n-2)$-cell. □

A.2 Local combinations of facets in strong substitutes LIPs

The balancing condition now allows us to infer, if a point is minimal in a certain way on a facet with a certain normal, the existence of certain other facets also passing through this point. We provide three results of this kind, Lemmas A.4, A.5 and A.6.

Lemma A.4. *Suppose F^i is an i-hod of strong substitutes LIP $\mathcal{L}_v$ with weight $w_v(F^i)$. Suppose that $\mathbf{r} \in F^i$ and there exists $j \in [n]$ with $j \neq i$ such that $\mathbf{r} - \epsilon \mathbf{e}^j \notin F^i$ for all $\epsilon > 0$. Suppose moreover that, if there is an i-hod F'^i, meeting F^i along an $(n-2)$-cell C such that $\mathbf{r} \in C$ and such that there exists $\mathbf{s} \in C$ and $\mathbf{s} - \epsilon \mathbf{e}^j \in F'^i$ for some $\epsilon > 0$, then $w' := w_v(F'^i) < w_v(F^i)$; otherwise write $w' = 0$.*

Then there exist:

(1) a j-hod $F^j \subseteq \{\mathbf{p} \in \mathbb{R}^n \mid p_i \geq r_i; p_j = r_j\}$;
(2) an ij-fin $F^{ij} \subseteq \{\mathbf{p} \in \mathbb{R}^n \mid p_i \leq r_i; (p_i - r_i) = p_j - r_j)\}$;
(3) an $(n-2)$-cell C, with affine span $\{\mathbf{p} \in \mathbb{R}^n \mid p_i = r_i; p_j = r_j\}$, in the boundary of F, F^j and F^{ij};

such that $\mathbf{r} \in C = F^i \cap F^j \cap F^{ij}$.

Moreover, if we write F'^j, F'^{ij} for facets, if they exist, with corresponding normal vectors but on the far side of the bounding $(n-2)$-cell C from their counterparts described above, and if we write respectively $w_v(F'^j) = 0$ and $w_v(F'^{ij}) = 0$ if such facets do not exist, then

$$w_v(F^i) - w_v(F'^i) = w_v(F^{ij}) - w_v(F'^{ij}) = w_v(F^j) - w_v(F'^j)$$

and in particular then $w_v(F^j) \geq w_v(F^i) - w'$ and $w_v(F^{ij}) \geq w_v(F^i) - w'$.

Proof. By minimality of $\mathbf{r}$ with respect to coordinate j, it follows that $\mathbf{r}$ lies on an $(n-2)$-cell C of F^i. By Lemma A.2, C cannot be Type 2 or 4, and by the assumption that $w' < w_v(F^i)$, Corollary A.3 shows that C cannot be Type 3. So, by Lemma A.2 again, C is Type 1. Since C bounds F^i with respect to coordinate j, it follows that C is Type 1 with indices i, j. Recall from Lemma A.2 that the other facets that may contain C in their boundary are j-hods or ij-fins.

Write F^j and F^{ij} and F'^i, F'^j, F'^{ij} for facets, if they exist, as described in the statement of the lemma. For each facet so described, write its weight as zero if it does not exist in $\mathcal{L}_v$. Then, since $\mathcal{L}_v$ is balanced around C, we know that (tracing a circle around C):

$$w_v(F^i)\mathbf{e}^i + w_v(F'^{ij})(\mathbf{e}^i - \mathbf{e}^j) + w_v(F^j)(-\mathbf{e}^j) + w_v(F'^i)(-\mathbf{e}^i) + w_v(F^{ij})(-\mathbf{e}^i + \mathbf{e}^j) + w_v(F'^j)\mathbf{e}^j = 0$$

By considering components in direction $\mathbf{e}^i$ and $\mathbf{e}^j$ in turn, we see that

$$w_v(F^i) - w_v(F'^i) = w_v(F^{ij}) - w_v(F'^{ij}) = w_v(F^j) - w_v(F'^j)$$

Since we assumed that $w_v(F^i) > w_v(F'_i) = w'$, we conclude that $w_v(F^{ij}) > 0$ and $w_v(F^j) > 0$, whence these facets do indeed exist. Since all weights are non-negative for a LIP, the final conclusion follows. □

Lemma A.5. *Suppose F^{ij} is an ij-fin of strong substitutes LIP $\mathcal{L}_v$ with weight $w_v(F^{ij})$. Suppose that $\mathbf{r} \in F^{ij}$ and $\mathbf{r} + \epsilon(\mathbf{e}^i + \mathbf{e}^j) \notin F^{ij}$ for all $\epsilon > 0$. Suppose moreover that, if* *there is an ij-fin F'^{ij}, meeting F^{ij} along an $(n-2)$-cell C such that $\mathbf{r} \in C$ and such that there exists $\mathbf{s} \in C$ and $_{+}\epsilon(\mathbf{e}^i + \mathbf{e}^j) \in F'^{ij}$ for some $\epsilon > 0$, then $w' := w_v(F'^{ij}) < w_v(F^{ij})$; otherwise write $w' = 0$.*

Then there exist:

(1) an i-hod $F^i \subseteq \{\mathbf{p} \in \mathbb{R}^n \mid p_i = r_i; p_j \geq r_j\}$;
(2) a j-hod $F^j \subseteq \{\mathbf{p} \in \mathbb{R}^n \mid p_i \geq r_i; p_j = r_j\}$;
(3) an $(n-2)$-cell C with affine span $\{\mathbf{p} \in \mathbb{R}^n \mid p_i = r_i; p_j = r_j\}$, in the boundary of F^i, F^j and F^{ij};

such that $\mathbf{r} \in C = F^i \cap F^j \cap F^{ij}$.

Moreover, if we F'^i, F'^j for facets, if they exist, with corresponding normal vectors but on the far side of the bounding $(n-2)$-cell C from their counterparts, and if we write respectively $w_v(F'^i) = 0$, $w_v(F'^j) = 0$ if such facets do not exist, then

$$w_v(F^i) - w_v(F'^i) = w_v(F^{ij}) - w_v(F'^{ij}) = w_v(F^j) - w_v(F'^j)$$

and in particular then $w_v(F^j) \geq w_v(F^i) - w'$ and $w_v(F^{ij}) \geq w_v(F^i) - w'$.

Proof. By maximality of $\mathbf{r}$ with respect to $\mathbf{e}^i + \mathbf{e}^j$, it follows that $\mathbf{r}$ lies on an $(n-2)$-cell C of F^{ij}. By the assumption that $w' < w_v(F^{ij})$, Corollary A.3 shows that C cannot be Type 3 or 4. So C is Type 1 or 2; indeed $\mathbf{r}$ may lie on $(n-2)$-cells of both Types, but it must lie on an $(n-2)$-cell of Type 1 with indices i, j as it is maximal with respect to $\mathbf{e}^i + \mathbf{e}^j$. So assume that C is of Type 1 with indices i, j. Recall from Lemma A.2 that the other facets that may contain C in their boundary are i-hods or j-hods.

Write F^j and F^i and F'^i, F'^j, F'^{ij} for facets, if they exist, as described in the statement of this lemma. For each facet so described, write its weight as zero if it does not exist in $\mathcal{L}_v$. Then, since $\mathcal{L}_v$ is balanced around C, we know that (tracing a circle around C):

$$w_v(F^i)\mathbf{e}^i + w_v(F'^{ij})(\mathbf{e}^i - \mathbf{e}^j) + w_v(F^j)(-\mathbf{e}^j) + w_v(F'^i)(-\mathbf{e}^i) + w_v(F^{ij})(-\mathbf{e}^i + \mathbf{e}^j) + w_v(F'^j)\mathbf{e}^j = 0$$

considering components in direction $\mathbf{e}^i$ and $\mathbf{e}^j$ in turn, we see that

$$w_v(F^i) - w_v(F'^i) = w_v(F^{ij}) - w_v(F'^{ij}) = w_v(F^j) - w_v(F'^j)$$

Since we assumed that $w_v(F^{ij}) > w_v(F'_{ij}) = w'$, we conclude that $w_v(F^i) > 0$ and $w_v(F^j) > 0$, whence these facets do indeed exist. Since all weights are non-negative, the final conclusion follows. □

Lemma A.6. *Suppose F^{ij} is an ij-fin of strong substitutes LIP $\mathcal{L}_v$ with weight $w_v(F^{ij})$. Suppose that $\mathbf{r} \in F^{ij}$ maximises $(\mathbf{e}^i - \mathbf{e}^k) \cdot \mathbf{r}'$ for $\mathbf{r}' \in F^{ij}$, where $k \in [n]$ and $k \neq i, j$. Suppose moreover that, if there is an ij-fin F'^{ij}, meeting F^{ij} along an $(n-2)$-cell containing $\mathbf{r}$ and containing a point $\mathbf{s}$ with $s_i > r_i$, then $w' := w_v(F'^{ij}) < w_v(F^{ij})$; otherwise write $w' = 0$.*

Then there exist:

(1) an ij-fin $F^{ik} \subseteq \{\mathbf{p} \in \mathbb{R}^n \mid (p_i - r_i) = (p_k - r_k) \leq (p_j - r_j)\}$;
(2) a jk-fin $F^{jk} \subseteq \{\mathbf{p} \in \mathbb{R}^n \mid (p_j - r_j) = (p_k - r_k) \leq (p_i - r_i)\}$;
(3) an $(n-2)$-cell C, with affine span $\{\mathbf{p} \in \mathbb{R}^n \mid (p_i - r_i) = (p_j - r_j) = (p_k - r_k)\}$, in the boundary of F^{ij}, F^{ik} and F^{jk};

such that $\mathbf{r} \in C = F^i \cap F^j \cap F^{ij}$.

Moreover, then $w_v(F^j) \geq w_v(F^i) - w'$ *and* $w_v(F^{ij}) \geq w_v(F^i) - w'$.

Proof. By maximality of $(\mathbf{e}^i - \mathbf{e}^k) \cdot \mathbf{r}'$ at $\mathbf{r}$, it follows that $\mathbf{r}$ lies on an $(n-2)$-cell C of F^{ij}. By the assumption that $w' < w_v(F^{ij})$, Corollary A.3 tells us that C cannot be of Type 3 or 4. So C is Type 1 or 2; indeed $\mathbf{r}$ may lie on $(n-2)$-cells of both Types. But if $\mathbf{r}$ does not lie on an $(n-2)$-cell of Type 2 with indices i, j, k then $(\mathbf{e}^i - \mathbf{e}^k) \cdot \mathbf{r}'$ cannot be maximised at $\mathbf{r}$. So assume that C is of Type 2 with indices i, j, k. Recall from Lemma A.2 that the other facets that may contain C in their boundary are ik-fin or jk-fins.

Write F^{ik} and F^{jk} for facets, if they exist, as described in the statement of the lemma, and $F'^{ij}, F'^{ik}, F'^{jk}$ for facets with corresponding normal vectors but on the far side of the bounding $(n-2)$-cell C from their counterparts. For each facet so described, write its weight as zero if it does not exist in $\mathcal{L}_v$. Then, since $\mathcal{L}_v$ is balanced around C, we know that (tracing a circle around C):

$$w_v(F^{ij})(\mathbf{e}^i - \mathbf{e}^j) + w_v(F'^{ik})(\mathbf{e}^i - \mathbf{e}^k) + w_v(F^{jk})(\mathbf{e}^j - \mathbf{e}^k) + w_v(F'^{ij})(\mathbf{e}^j - \mathbf{e}^i) + w_v(F^{ik})(\mathbf{e}^k - \mathbf{e}^i) + w_v(F'^{jk})(\mathbf{e}^k - \mathbf{e}^j) = 0$$

considering components in directions $\mathbf{e}^i$, $\mathbf{e}^j$ and $\mathbf{e}^k$ in turn, we see that

$$w_v(F^{ij}) - w_v(F'^{ij}) = w_v(F^{ik}) - w_v(F'^{ik}) = w_v(F^{jk}) - w_v(F'^{jk})$$

Since we assumed that $w_v(F^{ij}) > w_v(F'^{ij}) = w'$, and since all weights are non-negative, we conclude that $w_v(F^{ik}) > 0$ and $w_v(F^{jk}) > 0$, whence these facets do indeed exist. Since all weights are non-negative, the final conclusion follows. □

Write $B_\epsilon(\mathbf{r})$ for $\{\mathbf{p} \in \mathbb{R}^n \mid \|\mathbf{p} - \mathbf{r}\| < \epsilon\}$.

Corollary A.7. *Suppose* $\mathcal{L}_v$ *is a strong substitutes LIP with* ij*-fin* F^{ij} *with weight* $w = w_v(F^{ij})$*, and when we write* $\mathbf{r} = \bigwedge_{ij} F^{ij}$ *then* $\mathbf{r} \in \mathbf{H}$*. Write also* w' *for the maximal weight of a facet* $F' \subseteq \{\mathbf{p} \in \mathbb{R}^n \mid p_i \geq r_i\}$*, with* $\bigwedge_{ij} F' \leq \mathbf{r}$*. Assume* $w' < w$*.*

Then, for sufficiently small $\epsilon > 0$*, for all* $k \neq i, j$*, writing* $\mathbf{b}' = (\mathbf{r}, w - w')$ *we have* $(F^i_{\mathbf{b}'} \cap B_\epsilon(\mathbf{r}), w - w') \preceq (\mathcal{L}_v, w_v)$ *and* $(F^j_{\mathbf{b}'} \cap B_\epsilon(\mathbf{r}), w - w') \preceq (\mathcal{L}_v, w_v)$*;*

Proof. Follows from Lemma A.5. □

B Proofs of Results in the Text

B.1 Orders and Lattices

Proof of Lemma 3.8. Part (1) follows immediately from the fact that the Euclidean ordering gives $\mathbb{R}^n$ the structure of a lattice. For Part (2), it is straightforward to verify that $\geq_{ij}$ is reflexive, antisymmetric and transitive, and so that $(H^{ij}, \geq_{ij})$ is a poset. For any $\mathbf{r}, \mathbf{r}' \in H^{ij}_\alpha$, define $\mathbf{r} \vee_{ij} \mathbf{r}'$ and $(\mathbf{r} \wedge_{ij} \mathbf{r}')$ by

$$(\mathbf{r} \vee_{ij} \mathbf{r}')_i = \min(r_i, r'_i) \qquad (\mathbf{r} \wedge_{ij} \mathbf{r}')_i = \max(r_i, r'_i)$$
$$(\mathbf{r} \vee_{ij} \mathbf{r}')_j = \min(r_j, r'_j) \qquad (\mathbf{r} \wedge_{ij} \mathbf{r}')_j = \max(r_j, r'_j)$$
$$(\mathbf{r} \vee_{ij} \mathbf{r}')_k = \max(r_k - r_i, r'_k - r'_i) + \min(r_i, r'_i) \quad (\mathbf{r} \wedge_{ij} \mathbf{r}')_k = \min(r_k - r_i, r'_k - r'_i) + \max(r_i, r'_i).$$

We can check that these form a least upper bound and greatest lower bound for $\{\mathbf{r}, \mathbf{r}'\}$ in $\mathbb{R}^n$. It remains to check that they lie in H^{ij}_α. But if $\mathbf{r}, \mathbf{r}' \in H^{ij}_\alpha$ then $r_i = r_j + \alpha$ and $r'_i = r'_j + \alpha$ and hence $r_i \leq r'_i$ holds iff $r_j \leq r'_j$. Thus, if $(\mathbf{r} \vee_{ij} \mathbf{r}')_i = r_i$ then $(\mathbf{r} \vee_{ij} \mathbf{r}')_j = r_j$ and so $(\mathbf{r} \vee_{ij} \mathbf{r}')_i - (\mathbf{r} \vee_{ij} \mathbf{r}')_j = \alpha$. Other cases follow in the same way. □

Proof of Lemma 3.10. For (1) we prove the stronger statement, that every price complex cell for v is a lattice with respect to the Euclidean ordering. This follows by straightforward consideration of the possible bounds on such sets and the limited range of facet normals permitted in a strong substitutes LIP. Alternatively, it also follows from the well-known result that, for any strong substitutes valuation, the prices for which any given bundle is demanded form a lattice (see, e.g., Milgrom and Strulovici, 2009).

We now show (2) by explicitly identifying the infimum with respect to this order. Let $C = \arg\max\{r_i \mid \mathbf{r} \in F\}$ and let $\mathbf{r} = \bigwedge C$, where we take the infimum with respect to the standard order $\geq$. Observe that C is a price complex cell, being a face of F, and so $\mathbf{r} = \bigwedge C \in C \subsetneq F$ by the preceding paragraph. Now, if $\mathbf{r}' \in F$, we know $r'_i \leq r_i$ by definition of $\mathbf{r}$, and thus, since F has normal $\mathbf{e}^i - \mathbf{e}^j$, we also know that $r'_j \leq r_j$. Moreover, since $\mathbf{r} = \bigwedge C$ it follows that, for all $k \neq i, j$, the facet F also has an $(n-2)$ cell C^k passing through $\mathbf{r}$, such that there exists a vector $\mathbf{v}^k$ with $\mathbf{v}^k \cdot (\mathbf{r}^k - \mathbf{r}) = 0$ for all $\mathbf{r}^k \in C^k$, and such that $v^k_k \neq 0$. But by consideration of the possible Type of C^k (Definition A.1), we observe that in every case, it must follow that $r'_k - r'_i \geq r_k - r_i$. So $\mathbf{r}' \leq_{ij} \mathbf{r}$ for all $\mathbf{r}' \in F$ and hence $\mathbf{r} = \bigwedge_{ij} F$. □

Proof of Corollary 3.11. By Lemma 3.10 it is sufficient to show that all facets are bounded with respect to the suitable partial order. But consider prices $\mathbf{p}$ in an i-hod F. By Fact 3.2 we know that for all such $\mathbf{p}$ there are $\mathbf{x}, \mathbf{x}' \in D_v(\mathbf{p})$ such that $\mathbf{x}' - \mathbf{x} = w_v(F)\mathbf{e}^i$, and so in particular $\sum_i x_i \leq D - 1$. By definition of $A = D\Delta_{[n_0]}$, we know that, for all $j \in [n]$ we have $\mathbf{x} + \mathbf{e}^j \in A$. Now if $p'_j < (v(\mathbf{x} + \mathbf{e}^j) - v(\mathbf{x}))$ then $v(\mathbf{x} + \mathbf{e}^j) - \mathbf{p}' \cdot (\mathbf{x} + \mathbf{e}^j) > v(\mathbf{x}) - \mathbf{p}' \cdot \mathbf{x}$ and so $\mathbf{x} \notin D_v(\mathbf{p})$. Applying this for all j shows that F is bounded below with respect to $\leq_i$.

Now consider prices $\mathbf{p}$ in an ij-fin F. Again we know that for all such $\mathbf{p}$ there are $\mathbf{x}, \mathbf{x}' \in D_v(\mathbf{p})$ such that $\mathbf{x} - \mathbf{x}' = w_v(F)(\mathbf{e}^i - \mathbf{e}^j)$ and so $x_i > 0$. By definition of $A = D\Delta_{[n_0]}$, we know that $\mathbf{x} - \mathbf{e}^i \in A$. Now if $p'_i > v(\mathbf{x}) - v(\mathbf{x} - \mathbf{e}^i)$ then $v(\mathbf{x} - \mathbf{e}^i) - \mathbf{p}' \cdot (\mathbf{x} - \mathbf{e}^i) > v(\mathbf{x}) - \mathbf{p}' \cdot \mathbf{x}$ and so $\mathbf{x} \notin D_v(\mathbf{p})$. This provides the lower bound with respect to coordinate i. But we also know, since $\mathbf{x}, \mathbf{x} - \mathbf{e}^i \in A$ that also $\mathbf{x} - \mathbf{e}^i + \mathbf{e}^k \in A$ for all $k \in [n]$ with $k \neq i, j$. This time if $p'_k - p'_i < v(\mathbf{x} - \mathbf{e}^i + \mathbf{e}^k) - v(\mathbf{x})$ then $v(\mathbf{x} - \mathbf{e}^i + \mathbf{e}^k) - \mathbf{p}' \cdot (\mathbf{x} - \mathbf{e}^i + \mathbf{e}^k) > v(\mathbf{x}) - \mathbf{p}' \cdot \mathbf{x}$ and so $\mathbf{x} \notin D_v(\mathbf{p})$. Applying this for all suitable k and combining this with the result for i shows that F is bounded below with respect to $\leq_{ij}$. □

It is useful also to note at this point:

Lemma B.1.

(1) If $\mathbf{r} \leq_{ij} \mathbf{r}'$ and $r_i = r'_i$ then $r_j = r'_j$ and $\mathbf{r} \leq \mathbf{r}'$
(2) If $\mathbf{r} \leq \mathbf{r}'$ and $r_i = r'_i$ and $r_j = r'_j$ then $\mathbf{r} \leq_{ij} \mathbf{r}'$

Proof. Clear from Definition 3.7. □

B.2 Arithmetic of Pseudo-LIPs

First, we will save time by defining:

Definition B.2. Call (Π, w) a *full balanced complex* if Π is a rational polyhedral complexes of dimension n with support $\mathbb{R}^n$, and w is a balanced $\mathbb{Z}$-weighting on the facets ($(n-1)$-dimensional faces) of Π.

We define addition for full balanced complexes:

Definition B.3.

(1) Given full balanced complexes (Π^1, w^1) and (Π^2, w^2) define $(\Pi^1, w^2) \boxplus (\Pi^1, w^2)$ to be (Π, w) where Π is the polyhedral complex with cells $C^1 \cap C^2$ for $C^1 \in \Pi^1$, $C^1 \in \Pi^2$, and $w(F) := \sum_{F' \in \mathcal{F}^1} w^1(F') + \sum_{F' \in \mathcal{F}^2} w^2(F')$ in which $\mathcal{F}^i$ is the set of all facets of Π^i containing F, for $i = 1, 2$. (Note that $\mathcal{F}^i$ contains at most one element for $i = 1, 2$.)

(2) $(\Pi^1, w^1) \boxminus (\Pi^2, w^2) := (\Pi^1, w^1) \boxplus (\Pi^2, -w^2)$

Lemma B.4. *If* (Π^1, w^1) *and* (Π^2, w^2) *are full balanced complexes then so are* $(\Pi^1, w^1) \boxplus (\Pi^2, w^2)$ *and* $(\Pi^1, w^1) \boxminus (\Pi^2, w^2)$.

Proof. It is well-known that the set of intersections of cells from two polyhedral complexes forms a polyhedral complex (see, e.g. Grünbaum, 1967, Chapter 3 Section 3.2 Exercise 7). It clearly inherits support $\mathbb{R}^n$ from Π^1 and Π^2, whence it must be of dimension n. To show balancing, consider an $(n-2)$-cell G. If $G \subseteq G^i$ where G^i is an $(n-2)$-cell of Π^i for $i = 1$ or 2, then the balancing condition is satisfied around G^i by all facets of Π^i containing G^i. On the other hand, if the minimal cell of Π^i containing G is a facet F, then splitting F into two facets along the affine span of G yields a complex satisfying the balancing condition around this span of G. And if the minimal cell of Π^i containing G is an n-cell then there are no facets of Π containing G and the balancing condition is trivial. Putting these cases together and noting that weights are just added across the two complexes, yields the balancing condition for $(\Pi^1, w^1) \boxplus (\Pi^2, w^2)$.

Finally, if (Π^2, w^2) is a full balanced complex then so is $(\Pi^2, -w^2)$ and so the result for $(\Pi^1, w^1) \boxminus (\Pi^2, w^2)$ follows from that for $(\Pi^1, w^1) \boxplus (\Pi^2, -w^2)$. □

We now show that we can use the usual rules of addition and subtraction on full balanced complexes.

Lemma B.5. $\boxplus$ *and* $\boxminus$ *satisfy the usual rules of addition and subtraction, with* $(\mathbb{R}^n, 0)$ *playing the role of identity element. That is, for full balanced complexes* (Π^1, w^1), (Π^2, w^2) *and* (Π^3, w^3) *we have:*

(1) $(\Pi^1, w^1) \boxplus (\Pi^2, w^2) = (\Pi^1, w^2) \boxplus (\Pi^1, w^1)$
(2) $(\Pi^1, w^1) \boxplus ((\Pi^1, w^2) \boxplus (\Pi^3, w^3)) = ((\Pi^1, w^1) \boxplus (\Pi^1, w^2)) \boxplus (\Pi^3, w^3)$
(3) $(\mathbb{R}^n, 0) \boxplus (\Pi^1, w^1) = (\Pi^1, w^1) \boxplus (\mathbb{R}^n, 0) = (\Pi^1, w^1)$
(4) $(\Pi^1, w^1) \boxminus (\Pi^2, w^2) = (\mathbb{R}^n, 0) \boxminus ((\Pi^2, w^2) \boxminus (\Pi^1, w^1))$

Additionally, $(\Pi^1, w^1) \boxminus (\Pi^1, w^1) = (\Pi^1, 0)$.

Proof. (1) follows immediately from noting that the order of (Π^1, w^1) and $\Pi^2, w^2)$ is immaterial in Definition B.3. (2) is similarly clear when we note that both can be written as the polyhedral complex with cells $C^1 \cap C^2 \cap C^3$ where $C^i \in \Pi^i$, with $w(F)$ similarly adding the weights of all facets from any of these three complexes which contain F. (3) holds because $C^1 \cap \mathbb{R}^n = C^1$ for any cell of Π^1, and $(\mathbb{R}^n, 0)$ contains no facets to alter the weighting.

To show (4), re-write the right hand side as $(\mathbb{R}^n, 0) \boxminus ((\Pi^2, w^2) \boxplus (\Pi^1, -w^1))$. But this is equal to $(\mathbb{R}^n, 0) \boxplus (\Pi^3, w^3)$ where Π^3 is equal to the complex of intersections of cells in Π^2 and Π^1, and w^3 is defined on facets F of this complex by $w^3(F)$ being equal to (-1) times the weight of this facet in $(\Pi^2, w^2) \boxplus (\Pi^1, -w^1)$, that is, $-1 \times \left(\sum_{F' \in \mathcal{F}^2} w^2(F') + \sum_{F' \in \mathcal{F}^1} -w^1(F')\right) = \sum_{F' \in \mathcal{F}^1} w^1(F') - \sum_{F' \in \mathcal{F}^2} w^2(F')$, in which $\mathcal{F}^i$ is the set of all facets of Π^i containing F, for $i = 1, 2$. So we have shown that $(\Pi^3, w^3) = (\Pi^1, w^1) \boxminus (\Pi^2, w^2)$, which by application of (3) completes the proof.

Finally, since Π^1 is a polyhedral complex, the complex of $(\Pi^1, w^1) \boxminus (\Pi^1, w^1)$ is just Π^1, and it is clear that the weight of every facet is zero. □

These arithmetics for pseudo-LIPs and full balanced complexes are consistent:

Lemma B.6. *If $(\mathcal{L}^i, w^i)$ is the weighted pseudo-LIP of (Π^i, w^i) for $i = 1, 2$ then $(\mathcal{L}^1, w^1) \boxplus (\mathcal{L}^2, w^2)$ is the weighted pseudo-LIP of $(\Pi^1, w^1) \boxplus (\Pi^2, w^2)$ and $(\mathcal{L}^1, w^1) \boxminus (\mathcal{L}^2, w^2)$ is the weighted pseudo-LIP of $(\Pi^1, w^1) \boxminus (\Pi^2, w^2)$.*

Proof. The result for $\boxplus$ is clear from Definitions 3.14, Lemma 3.15 and Definition 3.17. The result for $\boxminus$ follows when we observe that $(\mathcal{L}^2, -w^2)$ is the weighted pseudo-LIP of $(\Pi^2, -w^2)$. □

Proof of Lemma 3.18. First observe that the weighted pseudo-LIP of $(\mathbb{R}^n, 0)$ is $(\emptyset, 0)$. Then the pseudo-LIP property for the $\boxplus$ and $\boxminus$ of two weighted pseudo-LIPs follow from Lemmas B.4 and B.6, and results (1)-(4) of Lemma 3.18 follow from Lemmas B.5 and B.6. It remains to show property (5). But, by Lemma B.6, and the final result of Lemma B.5, we know that $(\mathcal{L}^1, w^1) \boxminus (\mathcal{L}^1, w^1)$ is the weighted pseudo-LIP of $(\Pi^1, 0)$; since all facets of $(\Pi^1, 0)$ have weight 0 it follows that $(\mathcal{L}^1, w^1) \boxminus (\mathcal{L}^1, w^1) = (\emptyset, 0)$. □

B.3 Bids and Geometry

Lemma B.7 (Baldwin and Klemperer (2019) Lemma 2.9(2))**.** *The cells of the price complex are the intersections of closures of UDRs.*

Lemma B.8. *The valuation $v_{\mathbf{b}}$ has UDRs as follows:*

(1) $\mathbf{0}$ is demanded in $\{\mathbf{p} \in \mathbb{R}^n : p_j > r_j \text{ for } j = 1, \ldots, n\}$;

(2) $M\mathbf{e}^i$ is demanded in $\{\mathbf{p} \in \mathbb{R}^n : p_i < r_i,\ p_i - r_i < p_j - r_j \text{ for } j = 1, \ldots, n\}$.

Proof. (1). $\mathbf{0}$ is uniquely demanded at $\mathbf{p}$ iff for all $\sum_k c_k \mathbf{e}^k \in m\Delta_{[n_0]}$ we have

$$0 = v_{\mathbf{b}}(\mathbf{0}) > v_{\mathbf{b}}(\sum_k c_k \mathbf{e}^k) - \mathbf{p} \cdot \sum_k c_k \mathbf{e}^k = \sum_k c_k (r_k - p_k). \tag{8}$$

In particular, equation (8) is required to hold for all $j \in [n]$ such that we set $c_j = 1$ and we set $c_k = 0$ for $k \neq j$. But doing so reveals that $r_j < p_j$ for all $j \in [n]$. On the other hand, $r_j < p_j$ for $j = 1 \ldots, n$ is clearly sufficient for (8) to hold for all $\sum_k c_k \mathbf{e}^k \in m\Delta_{[n_0]}$.

(2). $M\mathbf{e}^i$ is uniquely demanded at $\mathbf{p}$ iff $M(r_i - p_i) = v_{\mathbf{b}}(M\mathbf{e}^i) - \mathbf{p}\cdot(M\mathbf{e}^i) > v_{\mathbf{b}}(\mathbf{0}) = 0$ and for all $\sum_k c_k\mathbf{e}^k \in m\Delta_{[n_0]}$ we have

$$M(r_i - p_i) = v_{\mathbf{b}}(M\mathbf{e}^i) - \mathbf{p}\cdot(M\mathbf{e}^i) > v_{\mathbf{b}}(\sum_k c_k\mathbf{e}^k) - \mathbf{p}\cdot\sum_k c_k\mathbf{e}^k = \sum_k c_k(r_k - p_k). \quad (9)$$

In particular, (9) must hold for any $j \neq i$ with $c_j = M$ and $c_i = 0$, whence $M(p_i - r_i) < M(p_j - r_j)$for all $j \neq i$, which is clearly sufficient for (9) to hold for all $\sum_i c_i\mathbf{e}^i \in m\Delta_{[n_0]}$.

Finally, the UDRs described already are dense in $\mathbb{R}^n$, so no other UDRs are possible. □

Proof of Lemma 4.2. Immediate from Lemmas B.7 and B.8. □

Lemma B.9. *Suppose that C is a polyhedron and that, for every cell C' in the boundary of C, we have $\mathbf{H}^\circ \cap C' \neq \emptyset$. Then C is uniquely determined by $C \cap \mathbf{H}^\circ$.*

Proof. Suppose that C has dimension k and consider it as as a full dimensional polyhedron lying in its affine span, $\mathbb{R}^k$. Such a set may be presented as the intersection of the set of half-spaces X of $\mathbb{R}^k$, such that in each case the intersection $\partial X \cap C = C'$, where ∂X is the boundary of the half-space X and C' is a top-dimensional proper face of C. But then C' is a cell in the boundary of C, and so by assumption $C' \cap \mathbf{H}^\circ \neq \emptyset$. It follows that ∂X, the affine span of C', is uniquely determined by $C' \cap \mathbf{H}^\circ$. Then X itself is the half-space in $\mathbb{R}^k$, positioned on the same side of C' as is C. Thus every such X is uniquely determined by $C \cap \mathbf{H}^\circ$, and hence so is C. □

Proof of Proposition 4.10. Consider a facet F of $\mathcal{L}_{v^1}$. We know that $F \cap \mathbf{H}^\circ \neq \emptyset$ and $F \cap \mathbf{H}^\circ \subseteq \mathcal{L}_{v^2}$. So $\mathcal{L}_{v^2}$ has a facet F^2 containing $F \cap \mathbf{H}^\circ$. Suppose that $F \cap \mathbf{H}^\circ \subsetneq F^2 \cap \mathbf{H}^\circ$. Then $\mathcal{L}_{v^1}$ has an $(n-2)$-cell C in the boundary of F, such that $C \cap \mathbf{H}^\circ$ is not in the boundary of F^2. So there is another facet F' of $\mathcal{L}_{v^1}$, not contained in the affine span of F, and with $F \cap F' = C$. Just as for F, we also know that $F' \cap \mathbf{H}^\circ \neq \emptyset$ and $F' \cap \mathbf{H}^\circ \subseteq \mathcal{L}_{v^2}$. It follows, since $F \cap F' \cap \mathbf{H}^\circ = C \cap \mathbf{H}^\circ \neq \emptyset$ by assumption, that indeed $C \cap \mathbf{H}^\circ$ is in the boundary of F^2. So indeed $F \cap \mathbf{H}^\circ = F^2 \cap \mathbf{H}^\circ$. Now apply Lemma B.9. □

Proof of Proposition 4.11. First choose H satisfying Assumption 4.9.

Fix $i \in [n]$. Observe that, by Assumption 4.9, every bundle $\mathbf{x} \in A$ is demanded at some $\mathbf{p}$ with $p_i < H$, and so in particular if $\mathbf{x}, \mathbf{x} + \mathbf{e}^i \in A$ then $v(\mathbf{x}) - \mathbf{p}\cdot\mathbf{x} \geq v(\mathbf{x} + \mathbf{e}^i) - \mathbf{p}\cdot(\mathbf{x} + \mathbf{e}^i) \Leftrightarrow v(\mathbf{x} + \mathbf{e}^i) - v(\mathbf{x}) \leq p_i < H$. Therefore if $p_i > H$ then $\mathbf{x} + \mathbf{e}^i \in D_v(\mathbf{p})$ is not possible, as $\mathbf{x}$ would be preferred.

Consider the hyperplane $X^i = \{\mathbf{p} \in \mathbb{R}^n \mid p_i = H\}$ and the UDRs of $\mathcal{L}_v$ with which it has nonzero intersection. For each bundle $\mathbf{x}$ demanded in such a UDR, we know by assumption that either $x_i = 0$ or $\mathbf{x} - x_i\mathbf{e}^i \notin A$. In either case write $\mathbf{x}' = \mathbf{x} - x_i\mathbf{e}^i$ and assign $\widehat{v}(\mathbf{x}') := v(\mathbf{x}) - x_iH$. Now consider $\mathcal{L}_{\widehat{v}}$.

Now, when $p_i = H$, we have $v(\mathbf{x}) - \mathbf{p}\cdot\mathbf{x} = v(\mathbf{x}') - \mathbf{p}\cdot\mathbf{x}'$, and so, unless $\mathbf{x} = \mathbf{x}'$ on the intersection of X^i and the UDR in which $\mathbf{x}$ is demanded, these bundles offer the highest indirect utility in the expanded domain, and so we now have an i-hod of $\mathcal{L}_{\widehat{v}}$. Moreover, $v(\mathbf{x}) - \mathbf{p}\cdot\mathbf{x} \geq v(\mathbf{x}') - \mathbf{p}\cdot\mathbf{x}'$ if and only if $p_i \leq H$. It follows that the bundles in A are still demanded for prices below X^i, while the bundles $\mathbf{x}'$ are the only bundles demanded for prices above X^i; in particular $\mathbf{x}$ is still demanded just beneath this new i-hod, whereas $\mathbf{x}'$ is demanded just above it.

Now consider UDRs of $\mathcal{L}_v$ that are separated by a single facet F at prices meeting X^i. The bundles $\mathbf{x}, \mathbf{y}$ demanded in these respective UDRs must satisfy $\mathbf{x} - \mathbf{y} = w\mathbf{d}$, where $w \in \mathbb{Z}$ and $\mathbf{d}$ is a strong substitutes vector. Moreover, $\mathbf{d} = \mathbf{e}^i$ is not possible by assumption on H. We saw that $\mathbf{x}' = \mathbf{x} - x_i\mathbf{e}^i$ and $\mathbf{y}' = \mathbf{y} - y_i\mathbf{e}^i$ are demanded in the corresponding UDRs of $\mathcal{L}_{\widehat{v}}$ just above X^i. There is no additional bundle demanded at any UDR also meeting X^i at $X^i \cap F$ because we have only at this point extended our valuation to incorporate bundles such as $\mathbf{x}', \mathbf{y}'$. So there is a facet F' of $\mathcal{L}_{\widehat{v}}$ meeting X^i along $F \cap X^i$, and by Fact 3.2 the normal to this facet is $\mathbf{x}' - \mathbf{y}' = \mathbf{x} - \mathbf{y} - (x_i - y_i)\mathbf{e}^i = w(\mathbf{d} - d_i\mathbf{e}^i)$; observe since $\mathbf{d} \neq \mathbf{e}^i$ and $\mathbf{d}$ is a strong substitutes vector that $\mathbf{d} - d_i\mathbf{e}^i$ is also a strong substitutes vector. Thus the additional facet we have introduced is the facet of a strong substitutes valuation.

Moreover, since $x'_i = 0$ for all the newly introduced bundles $\mathbf{x}'$, increasing p_i from those just above H to any higher value does not alter the trade-off between bundles. So the newly introduced facets, which all contain a vector in direction $\mathbf{e}^i$, continue in identical formation for all $p_i > H$, and there are no further facets to check. One may easily verify that now for every facet F of $\mathcal{L}_{\widehat{v}}$, if F is a j-hod then $(\bigwedge_j F)_i \leq H$, and if F is a jk-fin with j, k distinct then $(\bigwedge_{jk} F)_i \leq H$.

Having completed this construction for $i \in [n]$ we may apply it again for $i' \neq i$, observing on completion that every facet F of $\mathcal{L}_{\widehat{v}}$, if F is a j-hod then $(\bigwedge_j F)_i, (\bigwedge_j F)_{i'} \leq H$, and if F is a jk-fin with j, k distinct and $j, k \neq i$ then $(\bigwedge_{jk} F)_i (\bigwedge_{jk} F)_{i'} \leq H$.

But we also observe that, since $x_i = 0$ for any bundle $\mathbf{x}$ demanded under $\widehat{v}$ when $p_i > H$, there will be a UDR in which such a bundle is demanded when we take the intersection with $X^{i'}$ and hence we will extend the valuation to at least one bundle in which $x_i = x_{i'} = 0$. Continuing through all the coordinates we conclude that, when we have made all n extensions, bundle $\mathbf{0}$ will be in the domain.

Now we turn to the lower bounds. This is similar. This time we observe by Assumption 4.9, every bundle $\mathbf{x} \in A$ is demanded at some $\mathbf{p}$ with $p_i > -H$, and so, just as before, if $p_i < -H$ and $\mathbf{x}, \mathbf{x} + \mathbf{e}^i \in A$ then $\mathbf{x} \in D_v(\mathbf{p})$ is not possible, as $\mathbf{x} + \mathbf{e}^i$ would be preferred.

Consider the hyperplane $Y^i = \{\mathbf{p} \in \mathbb{R}^n \mid p_i = -H\}$ and the UDRs of $\mathcal{L}_v$ with which it has nonzero intersection. For each bundle $\mathbf{x}$ demanded in such a UDR, we know by assumption that either $\|\mathbf{x}\| = D$ or $\mathbf{x} + (D - \|\mathbf{x}\|)\mathbf{e}^i \notin A$. In either case write $\mathbf{x}' = \mathbf{x} + (D - \|\mathbf{x}\|)\mathbf{e}^i$ and assign $\widehat{v}(\mathbf{x}') := v(\mathbf{x}) + (D - \|\mathbf{x}\|)H$. Now consider $\mathcal{L}_{\widehat{v}}$.

Just as before, $v(\mathbf{x}) - \mathbf{p} \cdot \mathbf{x} \leq v(\mathbf{x}') - \mathbf{p} \cdot \mathbf{x}'$ if and only if $p_i \geq -H$. It follows that the bundles in A are still demanded for prices above Y^i, while the bundles $\mathbf{x}'$ are the only bundles demanded for prices below Y^i; in particular $\mathbf{x}$ is still demanded just beneath this above i-hod, whereas $\mathbf{x}'$ is demanded just below it.

Now consider UDRs of $\mathcal{L}_v$ that are separated by a single facet F at prices meeting Y^i. The bundles $\mathbf{x}, \mathbf{y}$ demanded in these respective UDRs must satisfy $\mathbf{x} - \mathbf{y} = w\mathbf{d}$, where $w \in \mathbb{Z}$ and $\mathbf{d}$ is a strong substitutes vector. Moreover, $\mathbf{d} = \mathbf{e}^i$ is not possible by assumption on H. We saw that $\mathbf{x}' = \mathbf{x} + (D - \|\mathbf{x}\|)\mathbf{e}^i$ and $\mathbf{y}' = \mathbf{y} + (D - \|\mathbf{x}\|)\mathbf{e}^i$ are demanded in the corresponding UDRs of $\mathcal{L}_{\widehat{v}}$ just below Y^i, and again we do not need to worry about demand of additional bundles at these prices. So there is a facet F' of $\mathcal{L}_{\widehat{v}}$ meeting Y^i along $F \cap Y^i$, and by Fact 3.2 the normal to this facet is $\mathbf{x}' - \mathbf{y}' = \mathbf{x} - \mathbf{y} - (\|\mathbf{y}\| - \|\mathbf{x}\|)\mathbf{e}^i = w(\mathbf{d} - \|\mathbf{d}\|\mathbf{e}^i)$. Now if $\mathbf{d} = \mathbf{e}^j$ for $j \neq i$ then $\mathbf{d} - \|\mathbf{d}\|\mathbf{e}^i = \mathbf{e}^j - \mathbf{e}^j$; if $\mathbf{d} = \mathbf{e}^j - \mathbf{e}^k$ for $j, k \in [n]$ with $j \neq k$ then $\mathbf{d} - \|\mathbf{d}\|\mathbf{e}^i = \mathbf{d} = \mathbf{e}^j - \mathbf{e}^k$; we saw the case

$\mathbf{d} = \mathbf{e}^i$ was not possible. So in every case $\mathbf{x}' - \mathbf{y}' = w\mathbf{d}$ where $\mathbf{d}$ is a strong substitutes vector, indeed the normal to an jk-fin, for some $j \neq k$.

Moreover, since $\|\mathbf{x}'\| = D$ for all the newly introduced bundles $\mathbf{x}'$, decreasing $\sum_i p_i$ from prices just below $-H$ to any lower value does not alter the trade-off between bundles. So the newly introduced facets, which all contain a vector in direction $\sum_i \mathbf{e}^i$, continue in identical formation as we move prices in this direction, and there are no further facets to check. One may easily verify that now for every facet F of $\mathcal{L}_{\widehat{v}}$, is a F is a j-hod then $(\bigwedge_j F)_i \geq H$, and if F is a jk-fin with j,k distinct then $(\bigwedge_{jk} F)_i \geq -H$.

Moreover, for any price $\mathbf{p}$ at which $p_j > H$ for all $j \neq i$, we know by the preceding construction that any $\mathbf{x} \in D_{\widehat{v}}(\mathbf{p})$ satisfies $x_j = 0$ for all $j \neq i$. So $\mathbf{x}' = D\mathbf{e}^i$, that is, we have included $D\mathbf{e}^i$ in the domain of the extended valuation.

Having completed this construction for $i \in [n]$ we may apply it again for $i' \neq i$, observing on completion that every facet F of $\mathcal{L}_{\widehat{v}}$, if F is a j-hod then $(\bigwedge_j F)_i, (\bigwedge_j F)_{i'} \leq H$, and if F is a jk-fin with j,k distinct and $j,k \neq i$ then $(\bigwedge_{jk} F)_i (\bigwedge_{jk} F)_{i'} \leq H$.

Continuing in this way for every $i \in [n]$ now gives us a valuation $\widehat{v}$, whose domain contains $\mathbf{0}$ and $D\mathbf{e}^i$ for all $i \in [n]$, and whose domain is contained in the simplex $D\Delta_{[n_0]}$. That is, the convex hull of this domain is just simplex $D\Delta_{[n_0]}$. It remains to use convexity to extend $\widehat{v}$ to $D\Delta_{[n_0]}$, which completes the proof. □

Proof of Lemma 4.12. For Part (1) and (2), consider first $\mathbf{b}$ with $m(\mathbf{b}) > 0$. Observe from Lemma 4.2 that, for any $i \in [n]$ we have $F^i_{\mathbf{b}} \cap \mathbf{H}^\circ \neq \emptyset$ if and only if $\ell(\mathbf{b}) \notin \overline{\partial}^j \mathbf{H}$ for all $j \in [n]$ and additionally $\ell(\mathbf{b}) \notin \underline{\partial}^i \mathbf{H}$. But in this case $0, i \in S(\ell(\mathbf{b}), \mathbf{H})$ and so $\mathcal{L}^{\mathbf{H}}_{\mathbf{b}}$ has an i-hod. Next, for any $i,j \in [n]$ with $i \neq j$, we have $F^{ij}_{\mathbf{b}} \cap \mathbf{H}^\circ \neq \emptyset$ if and only if $\ell(\mathbf{b}) \notin \underline{\partial}^i \mathbf{H} \cup \underline{\partial}^j \mathbf{H}$. But in this case $i,j \in S(\ell(\mathbf{b}), \mathbf{H})$ and so $\mathcal{L}^{\mathbf{H}}_{\mathbf{b}}$ has an ij-fin. Since the values of all bundles which are demanded under $v^{\mathbf{H}}_{\mathbf{b}}$ are the same as the values of these bundles under $v_{\mathbf{b}}$, it follows that these facets are located in the same places; the weights are trivially the same, so $(\mathcal{L}_{\mathbf{b}} \cap \mathbf{H}^\circ, w_{\mathbf{b}}) = (\mathcal{L}^{\mathbf{H}}_{\mathbf{b}} \cap \mathbf{H}^\circ, w^{\mathbf{H}}_{\mathbf{b}})$ in this case. The $m(r) < 0$ case now follows, by definition.

Recall that $D^{\mathbf{H}}_{\mathbf{b}}(\mathbf{p}) := D_{\mathbf{b}}(\mathbf{p}) \cap m(\mathbf{b}) \Delta_{S(\ell(\mathbf{b}),\mathbf{H})}$ as defined in Section 2.2. Now observe that if $m(\mathbf{b}) > 0$ then $D^{\mathbf{H}}_{\mathbf{b}}(\mathbf{p}) = D_{v^{\mathbf{H}}_{\mathbf{b}}}(\mathbf{p})$; if $m(\mathbf{b}) < 0$ then $D^{\mathbf{H}}_{\mathbf{b}}(\mathbf{p}) = -D^{\mathbf{H}}_{|\mathbf{b}|}(\mathbf{p})$ as usual. Now part (3) follows from Part (1) and Fact 3.2. Part (4) is trivially true since $S(\ell(\mathbf{b}), \mathbf{H}) = [n]_0$ and so $v^{\mathbf{H}}_{\mathbf{b}} = v_{\mathbf{b}}$ in that case. □

B.4 The Proof of Proposition 5.2

Proof of Lemma 5.5. Part (1) is clear by definition of $\langle \mathcal{L}_v \rangle$. To show Part (2), suppose that F is a facet of $\langle \mathcal{L}_v \rangle$ and that $F' \cap F$ is $(n-1)$-dimensional for some facet F' of $\mathcal{L}_v$. If $F \not\subseteq F'$ then there exists an $(n-2)$-cell C of F' such that $F \cap C$ is $(n-2)$-dimensional, but C is not in the boundary of F. But, by construction of $\langle \mathcal{L}_v \rangle$, it follows that there is no j-hod of $\mathcal{L}_v$ meeting F' at C; otherwise C would indeed be in the boundary of F. Now, by Corollary A.3, it follows that there exists another facet F'', with the same affine span as F', meeting C on the other side from F', and with the same weight in $\mathcal{L}_v$ as F'. Then we also have that $F'' \cap V$ is $(n-1)$-dimensional. Applying this argument repeatedly we see that $F \subseteq \mathcal{L}_v$, and that all facets with which it has $(n-1)$-dimensional intersection have the same weight. □

Lemma B.10. *Let $\mathcal{L}_v$ be a strong substitutes LIP with simplex domain, and, for $s \geq 0$, define $\mathcal{B}^s$ and $\mathcal{L}_{v^s}$ as in Definition 5.6. Assume that $(\mathcal{L}_{v^t}, w_{v^t})$ is a weighted strong substitutes LIP for $t \leq s$. Then:*

(1) If F is a hod of $\mathcal{L}_{v^s}$ then $F \subseteq \langle \mathcal{L}_v \rangle$
(2) If F is a facet of $\langle \mathcal{L}_v \rangle$ such that $F \cap \mathcal{L}_{v^s}$ is $(n-1)$-dimensional, then $F \subseteq \mathcal{L}_{v^s}$. Moreover, then all facets F' of $\mathcal{L}_{v^s}$ such that $F' \cap F$ is $(n-1)$-dimensional have the same weight.

Proof of Lemma B.10. We prove these by induction on s. The base cases, $s = 0$, concern $\mathcal{L}_{v^s} = \mathcal{L}_v$ and are shown in Lemma 5.5. Make the inductive hypothesis that Parts (1)–(2) hold for s, and that $(\mathcal{L}_{v^t}, w_{v^t})$ is a weighted strong substitutes LIP for $t \leq s+1$.

To show the inductive step of Part (1), recall that $(\mathcal{L}_{v^{s+1}}, w_{v^{s+1}}) := (\mathcal{L}_{\mathcal{B}^s}, w_{\mathcal{B}^s}) \boxminus (\mathcal{L}_{v^s}, w_{v^s})$. So if F is a hod of $\mathcal{L}_{v^{s+1}}$ then either $F \subset \langle F' \rangle$ where F' is a hod of $\mathcal{L}_{\mathcal{B}^s}$, or $F \subseteq \langle F'' \rangle$ where F'' is a hod of $\mathcal{L}_{v^s}$, or both. But for every $\mathbf{b} \in \mathcal{B}^s$, we know that $\ell(\mathbf{b})$ is a grid point for $\mathcal{L}_v$, and so for every hod F' of $\mathcal{L}_{\mathcal{B}^s}$, we know $\langle F' \rangle \subseteq \langle \mathcal{L}_v \rangle$. And by the inductive hypothesis, every hod F'' of $\mathcal{L}_{v^s}$ satisfies $F'' \subseteq \langle \mathcal{L}_v \rangle$, and thus $\langle F'' \rangle \subseteq \langle \mathcal{L}_v \rangle$. This proves the inductive step of Part (1).

For the inductive step of Part (2), consider a facet F of $\langle \mathcal{L}_v \rangle$. If $F \cap \mathcal{L}_{v^s}$ is $(n-1)$-dimensional, then we know by the inductive hypothesis that $F \subseteq \mathcal{L}_{v^s}$ and that the weight of $F \cap F'$ is the same for all facets F' of $\mathcal{L}_{v^s}$ with $(n-1)$-dimensional intersection with F. On the other hand, if $F \cap \mathcal{L}_{\mathcal{B}^s}$ is $(n-1)$-dimensional, then by construction of $\langle \mathcal{L}_v \rangle$ and $\mathcal{L}_{\mathcal{B}^s}$, we know that $F \subseteq \mathcal{L}_{\mathcal{B}^s}$ and that the weight of $F \cap F'$ is the same for all facets F' of $\mathcal{L}_{\mathcal{B}^s}$ with $(n-1)$-dimensional intersection with F. So now suppose that $F \cap \mathcal{L}_{v^{s+1}}$ is $(n-1)$-dimensional and recall again that $(\mathcal{L}_{v^{s+1}}, w_{v^{s+1}}) := (\mathcal{L}_{\mathcal{B}^s}, w_{\mathcal{B}^s}) \boxminus (\mathcal{L}_{v^s}, w_{v^s})$. If $F \not\subseteq \mathcal{L}_{v^s}$ then $\dim(F \cap \mathcal{L}_{v^s}) \leq n-2$ and so $F \cap \mathcal{L}_{\mathcal{B}^s}$ is $(n-1)$-dimensional, whence $F \subseteq \mathcal{L}_{\mathcal{B}^s}$. On the other hand, if $F \subseteq \mathcal{L}_{v^s}$, then since $(\mathcal{L}_{v^{s+1}}, w_{v^{s+1}})$ is a (positive-)weighted strong substitutes LIP, it also must follow that also $F \subseteq \mathcal{L}_{\mathcal{B}^s}$. Since in either case, all the facets it meets in top dimension in either $\mathcal{L}_{\mathcal{B}^s}$ or both $\mathcal{L}_{\mathcal{B}^s}$ and $\mathcal{L}_{v^s}$ have the same weight, and since by assumption at least one of these has positive weight, we conclude that $F \subseteq \mathcal{L}_{v^{s+1}}$, and all facets of $\mathcal{L}_{v^{s+1}}$ with which it has non-zero intersection, have the same weight. $\square$

Proof of Proposition 5.8. First we show that, if $\mathcal{L}_{v^t}$ is indeed a strong substitutes weighted LIP with simplex domain for $t \leq s$, then $\mathcal{B}^s$ is a finite collection of positive bids such that $(\mathcal{L}_{v^s}, w_{v^s}) \preceq (\mathcal{L}_{\mathcal{B}^s}, w_{\mathcal{B}^s})$.

Finiteness is clear as there are finitely many grid points, and there is only one bid at any grid point.

For any i-hod F of $\mathcal{L}_{v^s}$, since v^s has simplex domain, we know by Corollary 3.11 that $\bigwedge_i F \in F$. We know by Lemma B.10 that $F \subseteq \langle \mathcal{L}_v \rangle$. So F is contained in a union of facets of $\langle \mathcal{L}_v \rangle$, each with $(n-1)$-dimensional intersection with F, and in particular one such facet, F', must contain $\bigwedge F$. But, by Lemma B.10 again, then $F' \subseteq \mathcal{L}_{v^s}$. So F' is contained in a union of facets of $\mathcal{L}_{v^s}$; since, again, v^s has simplex domain, it follows that $\bigwedge_i F' \in F'$. Thus there is a bid $\mathbf{b} \in \mathcal{B}^s$ with $\ell(\mathbf{b}) = \bigwedge F'$ and $m(\mathbf{b}) \geq w_{v^s}(F)$. Since $\bigwedge F \in F'$ we know $\bigwedge F' \leq_i \bigwedge F$ and thus, by Corollary 4.3, we know $(F, w_{v^s}(\widehat{F})) \preceq (\mathcal{L}_{\mathbf{b}}, w_{\mathbf{b}})$.

Now let F be an ij-fin of $\mathcal{L}_{v^s}$ with weight w. Write $\mathbf{r} = \bigwedge_{ij} F$ (which, again, exists by Corollary 3.11) and write w' as in Corollary A.7. Observe that if $w' > 0$ then a bid $\mathbf{b} \in \mathcal{B}^s$ such that $(F', w') \preceq (\mathcal{L}_\mathbf{b}, w_\mathbf{b})$ will also satisfy $(F, w') \preceq (\mathcal{L}_\mathbf{b}, w_\mathbf{b})$. So if $w' \geq w$ then we can proceed by induction on the facets with the same affine span as F, and otherwise we need only exhibit a bid $\mathbf{b} \in \mathcal{B}^s$ such that $(F, w - w') \preceq (\mathcal{L}_\mathbf{b}, w_\mathbf{b})$.

By Corollary A.7, we know that if we write $\mathbf{b}' = (\mathbf{r}, w - w')$ then $(F^i_{\mathbf{b}'} \cap B_\epsilon(\mathbf{r}), w - w') \preceq (\mathcal{L}_{v^s}, w_{v^s})$. Let F' be a facet of $\langle \mathcal{L}_v \rangle$, contained in $\mathcal{L}_{v^s}$, containing $\mathbf{r}$, and with $(n-1)$-dimensional intersection with $F^i_{\mathbf{b}'}$; such a facet exists by Lemma B.10. Then, writing $\mathbf{r}' = \bigwedge F'$, we know that $\mathbf{r}' \leq \mathbf{r}$ since $\mathbf{r} \in F'$, and that $r'_i = r_i$. By Definition 5.6 we know there exists a bid $\mathbf{b} = (\mathbf{r}', m)$ where $m \geq w_{v^s}(F') \geq w - w'$. Also, by Corollary A.7, we know that $(F^j_{\mathbf{b}''} \cap B_\epsilon(\mathbf{r}), w - w') \preceq (\mathcal{L}_{v^s}, w_{v^s})$. As interiors of facets are disjoint this implies that $r'_j \geq r_j$, so since $\mathbf{r}' \leq \mathbf{r}$ we conclude that $r'_j = r_j$. Now by Lemma B.1 we know $\mathbf{r}' \leq_{ij} \mathbf{r}$. It follows that $(F, w - w') \preceq (\mathcal{L}_\mathbf{b}, w_\mathbf{b})$.

Now we show by induction on s that $\mathcal{L}_{v^s}$ is indeed a strong substitutes LIP with simplex domain. For the base case, recall that $\mathcal{L}_{v^0} := \mathcal{L}_v$, which we assume is a strong substitutes LIP. Make the inductive hypothesis that the statement is true for s; as shown above, then $\mathcal{B}^s$ is a finite collection of positive bids covering $\mathcal{L}_{v^s}$. But then $(\mathcal{L}_{v^{s+1}}, w_{v^{s+1}}) = (\mathcal{B}^s, w_{\mathcal{B}^s}) \boxminus (\mathcal{L}_{v^s}, w_{v^s})$ is a strong substitutes LIP by Lemma 3.20. Finally, since the domains of both v^s and $v^{\mathcal{B}^s}$ are simplices, one may easily establish by considering extreme points that the domain of v^{s+1} is a simplex. □

Proof of Lemma 5.10. We know that, for some $i \in [n]$, we have $\ell(\mathbf{b}) = \bigwedge F$ for some i-hod F of $\langle \mathcal{L}_v \rangle$ such that $F \subseteq \mathcal{L}_{\mathcal{B}^s}$, and such that $w_{v^s}(F') = m(\mathbf{b})$ for all facets F' of $\mathcal{L}_{v^s}$ such that $F' \cap F$ has dimension $(n-1)$. Conversely, any i-hod F' of $\mathcal{L}_v$ and containing $\ell(\mathbf{b})$ satisfies $F' \subseteq \langle \mathcal{L}_v \rangle$, so by minimality of $\ell(\mathbf{b})$ with respect to $\leq$, it follows that any such F' has $\ell(\mathbf{b}) = \bigwedge F'$. So, for sufficiently small $\epsilon > 0$, the union of all i-hods of $\mathcal{L}_{\mathcal{B}^s} \cap B_\epsilon(\ell(\mathbf{b}))$, is just equal to $F \cap B_\epsilon(\ell(\mathbf{b}))$, and all have weight $w(\mathbf{b})$.

But now, by Lemma A.4, for all $j \in [n]$ with $j \neq i$, there exists a j-hod satisfying $\ell(\mathbf{b}) \in F^j \subseteq \{\mathbf{p} \in \mathbb{R}^n \mid p_i \geq r_i; p_j = r_j\}$. By Corollary 5.7it follows from minimality of $\ell(\mathbf{b})$ in $\mathcal{B}^s$ that $\ell(\mathbf{b}) = \bigwedge F^j$. For the same reason, we know that there is no F'^j as in the statement of Lemma A.4 and so $w_{v^s}(F^j) = w_{v^s}(F) = w(\mathbf{b})$. But then $F^j \subseteq \langle \mathcal{L}_v \rangle$ by Lemma B.10 and so in particular there is a facet F'' in $\langle \mathcal{L}_v \rangle$ with $(n-1)$-dimensional intersection with F^j and containing $\ell(\mathbf{b})$, whence $F'' \subseteq \mathcal{L}_{v^s}$. Thus, just as above, the union of all j-hods of $\mathcal{L}_{\mathcal{B}^s} \cap B_\epsilon(\ell(\mathbf{b}))$, is just equal to $F'' \cap B_\epsilon(\ell(\mathbf{b}))$, and by Lemma A.4, all have weight $w(\mathbf{b})$.

Now, since $\ell(\mathbf{b})$ is minimal with respect to $\leq$ subject to $\mathbf{b} \in \mathcal{B}^s$, it follows that $(\mathcal{L}_{\mathcal{B}^{s+1}}, w_{\mathcal{B}^{s+1}}) = (\mathcal{L}_{\mathcal{B}^s}, w_{\mathcal{B}^s}) \boxminus (\mathcal{L}_{v^s}, w_{v^s})$ does not contain a hod containing $\ell(\mathbf{b})$. □

References

L. M. Ausubel and P. Milgrom. Ascending auctions with package bidding. *Frontiers of Theoretical Economics*, 1(1):1–42, 2002.

E. Baldwin and P. Klemperer. Tropical geometry to analyse demand. Mimeo. Available from elizabeth-baldwin.me.uk and www.paulklemperer.org, May 2014.

E. Baldwin and P. Klemperer. Understanding preferences: "demand types", and the existence of equilibrium with indivisibilities. *Econometrica*, 87:867–932, 2019.

E. Baldwin and P. Klemperer. Implementing Walrasian equilibrium – the language of product-mix auctions. In preparation, in preparation.

E. Baldwin, P. Goldberg, and P. Klemperer. Valid combinations of bids. Technical report, Oxford University, 2016a.

E. Baldwin, P. W. Goldberg, and P. Klemperer. Tropical intersections and equilibrium (day 2 slides). Hausdorff School on Tropical Geometry and Economics, 2016b.

E. Baldwin, P. W. Goldberg, P. Klemperer, and E. Lock. Solving strong-substitutes product-mix auctions, 2019.

E. Baldwin, M. Bichler, M. Fichtl, and P. Klemperer. Strong substitutes: Structural properties, and a new algorithm for competitive equilibrium prices, 2021.

M. Fichtl. Assignment messages cannot express all strong substitutes valuations. Technical report, Technical University of Munich, 2020.

M. Fichtl. Computing candidate prices in budget-constrained product-mix auctions. Working paper, Technical University of Munich, 2021.

B. Grünbaum. *Convex polytopes*, volume XVI of *Pure and Applied Maths.* Interscience Publishers, 1st edition, 1967.

F. Gul and E. Stacchetti. Walrasian equilibrium with gross substitutes. *Journal of Economic Theory*, 87(1):95–124, 1999.

J. W. Hatfield and P. R. Milgrom. Matching with contracts. *The American Economic Review*, 95(4):913–935, 2005.

A. S. Kelso and V. P. Crawford. Job matching, coalition formation, and gross substitutes. *Econometrica*, 50(6):1483–1504, 1982.

P. Klemperer. A new auction for substitutes: Central bank liquidity auctions, the U.S. TARP, and variable product-mix auctions. Working paper, Nuffield College, 2008.

P. Klemperer. The product-mix auction: A new auction design for differentiated goods. *Journal of the European Economic Association*, 8(2-3):526–536, 2010.

P. Klemperer. Product-mix auctions. Working Paper 2018-W07, Nuffield College, 2018.

D. Maclagan and B. Sturmfels. *Introduction to Tropical Geometry*, volume 161 of *Graduate Studies in Mathematics.* American Mathematical Society, Providence, RI, 2015.

G. Mikhalkin. Decomposition into pairs-of-pants for complex algebraic hypersurfaces. *Topology*, 43(5):1035–1065, 2004.

P. Milgrom. Assignment messages and exchanges. *American Economic Journal: Microeconomics*, 1(2):95–113, 2009.

P. Milgrom and B. Strulovici. Substitute goods, auctions, and equilibrium. *Journal of Economic Theory*, 144(1):212–247, 2009.

K. Murota. *Discrete Convex Analysis*, volume 10 of *Monographs on Discrete Mathematics and Applications*. SIAM, Philadelphia, 2003.

K. Murota and A. Shioura. M-convex function on generalized polymatroid. *Mathematics of Operations Research*, 24(1):95–105, 1999.

M. Ostrovsky and R. Paes Leme. Gross substitutes and endowed assignment valuations. *Theoretical Economics*, 10(3):853–865, 2015.

A. Shioura and A. Tamura. Gross substitutes condition and discrete concavity for multi-unit valuations: A survey. *Journal of the Operations Research Society of Japan*, 58 (1):61–103, January 2015.

D. M. Topkis. Minimizing a submodular function on a lattice. *Operations Research*, 26 (2):305–321, 1978.

N. M. Tran. The finite matroid-based valuation conjecture is false, 2020.