TROPICAL GEOMETRY TO ANALYSE DEMAND

Elizabeth Baldwin[*] and Paul Klemperer[†]

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http://users.ox.ac.uk/~wadh1180/ and www.paulklemperer.org

ABSTRACT

Duality techniques from convex geometry, extended by the recently-developed mathematics of tropical geometry, provide a powerful lens to study demand. We propose a new framework of "demand types", for categorising and understanding demand; our classification both incorporates existing definitions (such as substitutes, complements, "strong substitutes", etc.) and permits additional distinctions. We obtain easy-to-check necessary and sufficient conditions for the existence of a competitive equilibrium for indivisible goods. Our techniques also underpin Klemperer's (2008) *Product-Mix Auction*, introduced by the Bank of England in the financial crisis.

[*]Grantham Research Institute, London School of Economics, UK. e.c.baldwin@lse.ac.uk
[†]Nuffield College, Oxford University, UK. paul.klemperer@economics.ox.ac.uk

1 Introduction

This paper introduces a new way to think about economic agents' individual and aggregate demands for indivisible goods,[1] and provides a new set of geometric tools to use for this. Our model applies to agents who buy and/or sell, as well as to some matching models.

Economists mostly think about agents' demands by focusing on the direct utility functions. We instead begin by focusing on the geometric structure of the regions of price space in which an agent demands different bundles. Our crucial observation is that dividing price space in this way creates *precisely* the geometric structure which is studied in the recently-developed, non-Euclidean, branch of algebraic geometry called "tropical geometry".[2] We can therefore use the tools of convex and tropical geometry, such as the duality between the geometric structure of an agent's demand in price space and the same agent's demand in quantity space, to obtain new insights about demand. Moving backwards and forwards between the dual representations of demand in price space and quantity space improves our understanding of both.

For example, it is much easier to aggregate individual demands in price space, but translating aggregate demand back into quantity space allows a strong theorem that encompasses and extends many existing results about when a competitive equilibrium exists.

On the other hand, if we start from the (direct) valuation function in quantity space, our methods for translating to the dual in price space quickly reveal the key properties of demand. Many existing results in demand theory can be understood more readily, and developed more efficiently, using our tropical-geometric perspective than using traditional methods.

Individual demand

Examining these geometric structures also suggests a natural way of classifying demand: we say two valuations have the same "type" if certain sets of vectors associated with the geometric structures are the same.

Importantly, "types" are not merely a mathematically convenient way to categorise demands. The list of vectors in the demand type is a list of possible ways an agent's demand can change when prices change. So it specifies the possible comparative statics of demand, and thus much of what economists think important about valuations. Familiar concepts such as substitutes and complements are examples of "types".

For example, a purchaser of new spectacles who is interested in having spare pairs might always buy lenses and frames in the ratio 2:1, whatever the individual prices of the goods. So when running an auction in which goods' characteristics suggest natural

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[1]Baldwin and Klemperer (in preparation) show the relevance of our techniques to analysing *divisible* goods also, in contexts such as the Product-Mix Auction.

[2]This assumes, as is standard in the indivisible-goods literature, that preferences are quasilinear. Tropical geometry was developed by, among others, Mikhalkin (2004, 2005). We believe it has not previously been applied to economics. Goeree and Kushnir (2012) have recently used techniques of convex analysis (see, e.g., Rockafellar, 1970), on which tropical geometry builds, in a very different context. However, Danilov and Koshevoy, with their coauthors (see, in particular, Danilov et al., 2001, Danilov et al., 2003 and Danilov and Koshevoy, 2004) have developed methods of discrete convex analysis with closer connections to ours which we discuss later in the Introduction, and in detail in Sections 4 and 6.3.

rates of substitution, bidders can be asked to express preferences that come from the corresponding demand type.[3]

Our classification into "demand types" makes it elementary to check, using simple rules about the signs and magnitudes of the entries in these sets of vectors, whether a demand type is, for example, substitutes, or complements, or "strong substitutes" (Milgrom and Strulovici, 2009), or "gross substitutes and complements" (Sun and Yang, 2006), etc., etc. So this approach provides an easy test of the nature of preferences.

Moreover, understanding the nature of "types" allows us to develop general results about such preferences. The comparative statics encoded in a type refer to changes in demand between generic prices, at which demand is unique–this is the set of demand changes that the literature often restricts to, and is used, for example, in Ausubel and Milgrom's (2002) definition of substitutes. However, we develop a sufficient condition under which the set of changes that arise generically is the complete set of changes that can arise anywhere (the set of demand changes used in the definition of substitutes of Kelso and Crawford, 1982). We call demand types for which these sets are the same "complete"; our sufficient criterion for completeness can quickly and easily be checked using the determinants of sets of the vectors describing the demand. Furthermore, we show completeness is equivalent to a generalisation of Gul and Stacchetti's (1999) "single-improvement property".[4]

Examining the vectors of "demand types" also clarifies the relationships among categories of demands. For example, it makes clear why the conditions for all of three or more indivisible goods to be (ordinary) substitutes are far more restrictive than the conditions for all of them to be complements–although these conditions are of course symmetric in the two-good case.

Aggregate demand and the existence of equilibrium

Understanding the *aggregate* demand of multiple agents allows us to develop a simple necessary and sufficient condition on preferences that guarantees the existence of a competitive equilibrium for indivisible goods. The condition is the *same* condition as that providing "completeness" above. So we can quickly see whether any demand structure guarantees equilibrium existence, simply by checking the determinants of sets of the vectors describing the demand. For example, we exhibit a demand type involving only complementary relationships between goods, for which equilibrium always exists.[5]

Furthermore, it is straightforward in our framework that properties such as the existence of equilibrium are preserved under (unimodular) basis changes of these same vectors.[6] Using this observation reveals when important properties of demands are the same.

The same observation demonstrates that the existence of equilibrium is – contrary to popular belief – *not* associated with substitutes relationships. Not only are there demand types which involve only complements and for which equilibrium always exists,

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[3]Indeed the version of the Product-Mix Auction now being used by the Bank of England has one-for-one substitution built into its design (see below and Klemperer, 2008, 2010).

[4]This has previously been extended to the "gross substitutes and complements" case by Sun and Yang (2009).

[5]The demand type is fundamentally different from (i.e., not simply a basis change of) strong substitutes, unlike, e.g., "gross substitutes and complements"—see below.

[6]A unimodular matrix is an integer square matrix with integer inverse (i.e., with determinant ±1).

but every demand type for which equilibrium is guaranteed can be obtained as a basis change of a demand type involving only complementary relationships – and this is *not* true of substitutes.

Understanding when equilibrium exists with complementary relationships also allows us to obtain new results about matching models with many-player matches since, we show, stable matchings in such models correspond to the competitive equilibria of "markets" in which people are complements to those they can be matched with.

Auctions

Finally, our understanding of the convex- and tropical-geometric structure of agents' preferences facilitates the analysis of "Product-Mix Auctions" (Klemperer, 2008, 2010; Baldwin and Klemperer, in preparation).[7] In these auctions–introduced by the Bank of England in response to the 2007 Northern Rock bank run and the subsequent financial crisis–bidders offer prices for alternative bundles of goods, so their bids can be represented geometrically as sets of points in multi-dimensional price space.[8] Our geometric techniques tell us what kinds of bids are needed to represent different kinds of preferences, what "coherent" bids look like, how to efficiently solve for equilibrium (and when it exists[9]), etc.[10]

Organisation of this paper

We begin, therefore, by explaining the basic concepts of tropical geometry. Section 2 describes the properties of a "tropical hypersurface", a geometric object which contains precisely those points at which the agent is indifferent between two or more bundles. Moreover, we observe that any geometric structure of this kind corresponds to a valuation function, so we can develop our understanding of demand by working directly with these geometric objects; we believe this is the first paper to do this.

A tropical hypersurface is composed of linear pieces known as "facets" which separate the regions of price space in which an agent's demand is for some specific unique bundle.

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[7] Product-mix auctions are "one-shot" auctions for allocating heterogeneous goods. Their equilibrium allocations and prices are similar to those of Simultaneous Multiple-Round Auctions in private-value contexts, but they permit the bid-taker to express richer preferences, are more robust against collusive and/or predatory behaviour, and are, of course, much faster.

[8] Bids are made as lists of coordinates in implementations like the Bank of England's; the Bank itself (the bid-taker) depicts these bids, and also its own preferences, geometrically.

Shortly after introducing the auction, an Executive Director of the Bank of England noted that it was "a world first in central banking", and hailed it as "potentially a major step forward in practical policies to support financial stability". And after regularly using the new design, and having auctioned over £100 billion in funds, the Governor of the Bank (Mervyn King) told *The Economist* that the Product-Mix Auction "is a marvellous application of theoretical economics to a practical problem of vital importance to financial markets". (See Bank of England, 2010, 2011, Milnes, 2010, Fisher, 2011, Fisher et al., 2011, and The Economist, 2012.) In principle, of course, funds are almost continuously divisible, but we can apply our same indivisible-good duality techniques.

[9] Klemperer (2008, 2010) explains that equilibrium always exists in the Bank of England's auction and simple extensions of it. It follows from Theorem 6.8 below that equilibrium exists for broader classes of preferences (see Baldwin and Klemperer, in preparation).

[10] Expressing even richer preferences, and over more goods, than the Bank of England's current implementation permits may in some circumstances be important to this or other Central Banks who have shown interest in using the auction, or for other applications such as the sale of related products by a manufacturer, the purchase of electricity generated in different locations, the trading of permits for emission reductions relating to different kinds of deforestation, etc. Our geometric methods also permit easy alternative ways of representing preferences as bids.

Section 3 explores duality in our context. The same set of vectors that are orthogonal to the facets of the tropical hypersurface also generates the surface of the agent's valuation function in quantity space (strictly, it generates the convex hull of that surface). So there is a precise correspondence between classes of tropical hypersurfaces (in price space) and subdivisions of "Newton polytopes" (in quantity space).

Section 4 focuses further on the structure of individual demand, by defining a "type" of demand by the same set of vectors as above. Since these vectors describe the ways in which the bundles demanded by the agent change with prices, they identify the key characteristics of demand. Our representation permits the easy proof and interpretation of existing results in the theory of demand. We also introduce generalisations of Gul and Stacchetti (1999)'s "single improvement property" as an alternative way to understand "types" and their properties.

Section 5 focuses on the important class of "complete" demand types, for which the set of changes in demand from all possible prices is the same as the set of changes in demand from the (generic) prices at which demand is unique. We show that a demand type on n goods is always complete if it is "unimodular": every subset of n of the vectors that define the type has determinant 0, +1, or -1 (plus an additional condition if the demand type's set of vectors do not span $\mathbb{R}^n$). We also connect "completeness" to the "single improvement property".

Section 6 turns to the analysis of *aggregate* demand. Working in price space makes aggregating agents' valuations easy. The tropical hypersurface of aggregate demand is simply the union of the tropical hypersurfaces of the individual demands – so it is also obvious that an aggregate valuation has demand of a certain "type" if and only if all individual agents do too.

Whether or not *equilibrium* exists depends on what might happen when more than one agent is indifferent between bundles. So we work with intersections of tropical hypersurfaces: the theory of tropical intersection multiplicities inspires our proofs that a competitive equilibrium always exists if all agents have concave demands of a given type, if and only if the *same* condition as discussed in the previous section–unimodularity–is satisfied by this type.

Although Bikhchandani and Mamer (1997) and Ma (1998) have previously given conditions for existence of equilibrium for a set of agents, their conditions are imposed upon the *aggregate* behaviour of all the agents in the economy, so must be checked against every possible combination of agents which, in many cases, seems neither practical nor to provide great insight into why agents' demands do or do not permit equilibrium. By contrast, here we give a necessary and sufficient criterion on the conditions that, when imposed upon each agent *individually*, guarantee competitive equilibrium.

The theorem we present tells us first, that if each valuation individually has a certain property, and that property satisfies our *sufficiency* criterion, then competitive equilibrium always exists. That is, we provide a class of results, each result stating that competitive equilibrium always exists when every valuation has a certain property. An example of such a result is that competitive equilibrium always exists if all valuations satisfy the "strong substitutes" property of Milgrom and Strulovici (2009).[11] We show

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[11]This example is not a new result, but it follows immediately from our theorem, as do several other existing results including Sun and Yang's (2006) result about the existence of equilibrium in their "two-group gross substitutes and complements" economy (which, like Milgrom and Strulovici's result,

that other, new, examples are easy to generate. For example, we exhibit a family of valuations on four goods involving *only* complementarities, and which is *not* a unimodular basis change of strong substitutes, and for which equilibrium always exists.

Our theorem also gives *necessity.* So we can quickly check, for any demand type, whether equilibrium will always exist if agents' valuations are of that demand type, or whether equilibrium must sometimes fail. For example, it follows easily that with three or fewer goods, a competitive equilibrium always exists if *and only if* goods are "strong substitutes", or are a (unimodular) basis change of strong substitutes.[12] Although this is not true in higher dimensions (see previous paragraph), competitive equilibrium cannot be guaranteed for most demand types. A benefit of our geometric approach is that our necessity result immediately provides an example of failure of equilibrium from every instance of failure of our criterion.

This theorem is closely related to the work in a remarkable series of papers by Danilov and Koshevoy and their co-authors. In particular, Danilov et al. (2001) provide a sufficient condition for equilibrium, which is mathematically dual to our *sufficient* condition. However, by adding an understanding of 'demand types', we make clear how these ideas may be applied.[13] Our development of that theory illuminates the economic nature of individual level conditions which guarantee equilibrium, and also makes clear the sense in which unimodularity is also *necessary* for equilibrium, which is not proved in Danilov et al. (2001).[14]

Thus our results in Sections 6.1-6.4 show how our geometric methods improve on existing results in determining whether any set of bidders whose preferences are drawn from a class of value functions (i.e., a "demand type") are guaranteed to always have equilibrium, whatever the bidders' precise value functions, and whatever the market supply. However, we show in Section 6.5 that tropical intersection theory also improves on existing techniques for understanding whether combinations of specific value functions also always yield equilibria.

We also show how our analysis can obtain new results about stable matchings in models with many-player matches.

Finally, we observe that since it is straightforward to "add" tropical hypersurfaces in price space, a natural and easy way to compute aggregate demand from agents' direct utility functions is by first computing each agent's tropical hypersurface. This is

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is a generalisation of Kelso and Crawford's (1982) results); Hatfield et al. (2013)'s result about when a stable outcome is not guaranteed in a trading network; and Teytelboym (2014)'s proof of equilibrium existence in his model of contracts and trading on networks; as well as extensions of many of these results.

[12]Thus our necessity theorem identifies the *classes* of valuation functions for which competitive equilibrium is guaranteed. This contrasts with "necessity" results of the kind given in several of the works of Footnote 11, which show only that equilibrium always exists if all agents' valuation functions have a certain property, but may fail if exactly one valuation function does not have this property.

[13]Thus, although the papers mentioned in Footnote 11 are subsequent to this work, these papers do not present their results as applications of it, since its relevance was not clear.

[14]We discuss the relationships to, and other distinctions from, Danilov and Koshevoy and their co-authors' work in Sections 4 and 6.3. We develop economic implications further than they do, aided by our concept of "demand types", whose importance is especially clear in price space (Danilov et al., 2001, 2003, 2008, work almost exclusively in to quantity space). Moreover, though our techniques are novel, they are more straightforward than theirs. However, their work deserves *far* more attention than it has thus far received–it seems to have been largely overlooked by the existing literature (such as that in Footnote 11).

essentially a generalisation of the point that it is easy to compute total demand from individuals' bids in a Product-Mix Auction. However, we defer substantive discussion of the application of tropical geometry to Product-Mix Auctions to Baldwin and Klemperer (in preparation). So we conclude in Section 7.

2 Representing Demand in Tropical Geometry

2.1 Assumptions and Motivation

There are n goods, which come in indivisible units. Each agent has a valuation function $u : A \to \mathbb{R}$ over a finite set $A \subsetneq \mathbb{Z}^n$ of possible bundles, which we call the *domain* of the valuation. We permit negative bundles to allow consideration of sellers as well as buyers. Agents have quasilinear preferences (and so, for example, no budget constraints). The price vector is $\mathbf{p} \in \mathbb{R}^n$, so different units of the same good always have the same price.[15] So the agent's demand set is

$$D_u(\mathbf{p}) := \arg\max_{\mathbf{x} \in A}\{u(\mathbf{x}) - \mathbf{p}.\mathbf{x}\}.$$

We are interested in how $D_u(\mathbf{p})$ varies with $\mathbf{p}$. It is of course constant while it is single-valued. All the action takes place at those $\mathbf{p}$ at which more than one bundle is demanded. So this set of prices is our principal object of study. We write this set of prices as[16]

$$\mathcal{T}_u := \{\mathbf{p} \in \mathbb{R}^n \mid \#D_u(\mathbf{p}) > 1\} . \tag{1}$$

The object $\mathcal{T}_u$ (with some additional structure–see Definition 2.3) is a convex-geometric object, known as a 'tropical hypersurface' (TH) in the new sub-discipline of algebraic geometry known as tropical geometry.[17] We believe this paper represents the first use of this structure in economics. The next two Sections (Sections 2 and 3) translates the relevant mathematics literature into our economic context.

A simple example is shown in Figure 1. The agent's valuations are $u(0,0) = 0$, $u(1,0) = 5$ and $u(0,1) = 4$. So its demand is for precisely one of these bundles in each of the three regions labelled, but switches from one bundle to another along the lines drawn.

The following subsections describe properties of THs, and also how the structure of the agents' demands can be recovered from them. The 'tropical' concepts may at first sound alien, but many aspects of working in price space should in fact be very natural to economists.

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[15]We can, of course, model different units of a homogeneous good which are priced independently, by simply treating them as different goods.

[16]We follow the mathematical literature in this slight abuse of notation.

[17]See Mikhalkin (2004) and others. In fact, Mikhalkin (2004) takes the tropical hypersurface associated to u to be the non-smooth locus of $\mathbf{p} \mapsto \max_{\mathbf{x} \in A}\{\mathbf{x}.\mathbf{p} - u(\mathbf{x})\}$. Thus our tropical hypersurfaces are 'upside down' compared with his. Mikhalkin's convention is not universal; Maclagan and Sturmfels (2009) take the non-smooth locus of $\mathbf{p} \mapsto \min_{\mathbf{x} \in A}\{u(\mathbf{x}) + \mathbf{x}.\mathbf{p}\}$, which defines tropical hypersurfaces the 'same way up' as ours, albeit shifted. Our convention seems the natural one from an economic point of view: we maximise surplus, that is, the value of a bundle minus its cost.
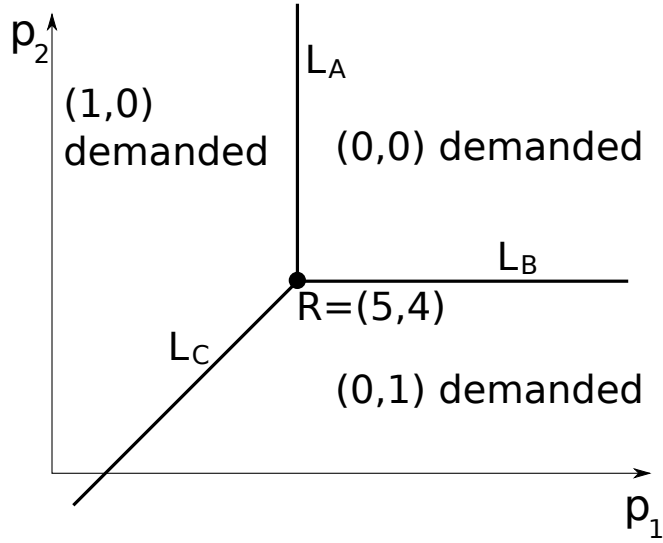
Figure 1: A simple tropical hypersurface (TH). The bundle demanded on each side of the TH is labelled.

2.2 The Tropical Hypersurface: associating geometric objects with demand

We start by considering the local structure of a TH. Given a price $\mathbf{p}$ and its demand set $D_u(\mathbf{p})$, we ask for what other prices $\mathbf{p}'$ the demand set is the same, or closely related.

Definition 2.1.

1. The *cell interior* of the TH $\mathcal{T}_u$ at a price $\mathbf{p}$ consists of points $\mathbf{p}'$ such that $D_u(\mathbf{p}) = D_u(\mathbf{p}')$.[18] A subset of $\mathcal{T}_u$ is a cell interior if it is the cell interior at some point in $\mathcal{T}_u$.

2. A subset of $\mathcal{T}_u$ is a *cell* if it the closure of a cell interior of $\mathcal{T}_u$.

3. The *affine span* of a cell of $\mathcal{T}_u$ is the smallest affine space containing the cell.[19]

4. The *boundary* of a cell of $\mathcal{T}_u$ consists of those points in the cell that are not in its cell interior.

Note that the cell interior is the largest set that is both contained in the cell and open in the affine span of the cell.[20]

We call a cell of dimension k a k-*cell*,[21] and call an $(n-1)$-cell a *facet*.

Figure 1 illustrates these concepts. The three line-segments L_A, L_B and L_C in the figure do *not* include the point R. Each of these line-segments is a cell interior: $D_u(\mathbf{p}) =$

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[18]Note that cells are subsets of the TH $\mathcal{T}_u$, and not, as one might intuitively guess from looking at Figure 1, the open areas around the sides of the TH; these are the 'unique demand regions'.

[19]Recall that an affine space in $\mathbb{R}^n$ is a parallel shift of a linear subspace, that is, a set $\{\mathbf{v}+\mathbf{c} \mid \mathbf{v} \in U\}$ for some linear subspace $U \leq \mathbb{R}^n$ and some fixed vector $\mathbf{c}$.

[20]See the equations for the three objects, given below. One might strictly refer to the 'cell interior' as the *relative interior* of the cell.

[21]To be precise, the dimension of a cell is the dimension of its affine span.
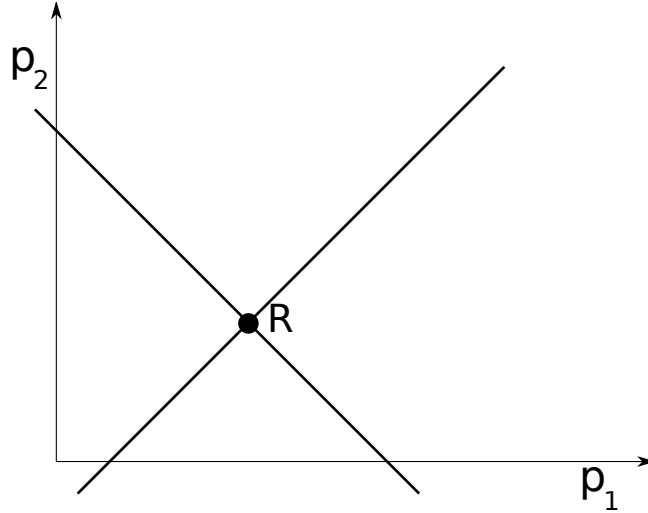
Figure 2: Cell interiors do not intersect; the line segments on either side of R are distinct cells.

$\{(0,0),(1,0)\}$ in L_A, $D_u(\mathbf{p}) = \{(0,0),(0,1)\}$ in L_B, and $D_u(\mathbf{p}) = \{(1,0),(0,1)\}$ in L_C. The point R is also a cell interior: $D_u(R) = \{(0,0),(1,0),(0,1)\}$. The corresponding cells are the unions of these cell interiors with their limit points: $L_A \cup R$ is thus a cell, and indeed a facet; so are $L_B \cup R$ and $L_C \cup R$. Finally, R itself is a 0-cell.

The price R is also the boundary of each of the 1-cells $L_A \cup \{R\}, L_B \cup \{R\}, L_C \cup \{R\}$. (The 0-cell R has no boundary.) Note that the price R is contained in four cells, but each price in the TH is contained in precisely one cell interior. Finally, the affine span of any cell is the set of all prices at which the agent is indifferent between all the bundles in the cell, so the affine spans of $L_A \cup R$, $L_B \cup R$, and $L_C \cup R$, are the entire lines containing those line-segments, while the affine span of R is the point R itself.

It is immediate that:

I There are finitely many distinct cells, and the TH is the union of these.

II The cell interiors do not intersect.

Figure 2 illustrates the latter point: although the TH is 'two line segments crossing at a point', it has *four* 1-cells with distinct interiors (and also a single 0-cell at R). Furthermore Definition 2.1 implies that for a price $\mathbf{p}'$ to be in the cell interior corresponding to a set of bundles $D_u(\mathbf{p})$, the agent must be indifferent between those bundles, that is, $\mathbf{p}'.(\mathbf{x} - \mathbf{x}') = u(\mathbf{x}) - u(\mathbf{x}')$ for all $\mathbf{x}, \mathbf{x}' \in D_u(\mathbf{p})$, and the agent must strictly prefer these bundles to all others, that is, $\mathbf{p}'.(\mathbf{x} - \mathbf{x}'') < u(\mathbf{x}) - u(\mathbf{x}'')$ for all $\mathbf{x} \in D_u(\mathbf{p})$ and $\mathbf{x}'' \in A \backslash D_u(\mathbf{p})$. The cell corresponding to this cell interior contains its limit points, so a price $\mathbf{p}'$ is in the cell if the bundles in $D_u(\mathbf{p})$ are *weakly* preferred to all others at this price; that is, we weaken the strict inequality above to a weak inequality (while maintaining the indifference between bundles in $D_u(\mathbf{p})$).[22] So a cell is the intersection of a finite number of half-spaces (sets $\{\mathbf{p}' \in \mathbb{R}^n \mid \mathbf{p}'.\mathbf{v} \le \alpha\}$ for some $\mathbf{v} \in \mathbb{R}^n$ and some $\alpha \in \mathbb{R}$). Thus:

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[22]It follows that we could alternatively define a cell as those points $\mathbf{p}'$ such that $D_u(\mathbf{p}) \subseteq D_u(\mathbf{p}')$ for some demand set $D_u(\mathbf{p})$.

III Each cell is a closed convex polyhedral set in $\mathbb{R}^n$.

The affine span of the cell corresponding to $D_u(\mathbf{p})$ is simply those $\mathbf{p}'$ such that $\mathbf{p}'.(\mathbf{x} - \mathbf{x}') = u(\mathbf{x}) - u(\mathbf{x}')$ for all $\mathbf{x}, \mathbf{x}' \in D_u(\mathbf{p})$. So the affine span of the cell is parallel to a linear subspace of $\mathbb{R}^n$, and, since $\mathbf{x}, \mathbf{x}' \in \mathbb{Z}^n$, we have:

IV The slope of the affine span of each cell is rational.

Finally, the boundary of the cell corresponding to $D_u(\mathbf{p})$ is those $\mathbf{p}'$ such that at least one of the weak inequalities $\mathbf{p}'.(\mathbf{x} - \mathbf{x}'') \leq u(\mathbf{x}) - u(\mathbf{x}'')$ for $\mathbf{x} \in D_u(\mathbf{p})$, $\mathbf{x}'' \in A \backslash D_u(\mathbf{p})$ holds with equality. Such points therefore also lie in a lower dimensional cell, so by restricting a suitable choice of inequalities to be equalities, we have:

V The boundary of a k-cell is a union of a finite number of $(k-1)$-cells.

On the other hand, any $(k-1)$-cell lies in the boundary of some k-cell (since, from the equations defining any $(k-1)$-cell, we can obtain the equations defining some k-cell by weakening one or more of the equalities). It follows that a TH is contained in the union of its facets.

We can therefore conclude that every TH for demand over n distinct goods can be understood as an $(n-1)$-dimensional *rational polyhedral complex*:

Definition 2.2 (Mikhalkin, 2004, Definitions 1 and 2). A subset $\Pi \subsetneq \mathbb{R}^n$ is a *rational polyhedral complex* if it is a finite union of closed sets in $\mathbb{R}^n$ called cells which satisfy properties **I**-**V** above. Π is *k-dimensional* if it is contained in the union of its k-cells.

By definition, demand in the complement of a TH is unique. We call a connected component of the complement of a TH a *unique demand region* (UDR). Demand is constant in each UDR, since the bundle demanded cannot change without the price crossing the TH. But to understand how demand changes as we move between UDRs, we need one additional type of information: 'weightings' on the facets.

Let F be a facet and let bundles $\mathbf{x}$ and $\mathbf{x}'$ be demanded in the UDRs on either side. So at prices $\mathbf{p} \in F$, the agent is indifferent between $\mathbf{x}$ and $\mathbf{x}'$, that is, $u(\mathbf{x}) - \mathbf{p}.\mathbf{x} = u(\mathbf{x}') - \mathbf{p}.\mathbf{x}'$. The crucial point is that because $\mathbf{p}.(\mathbf{x}' - \mathbf{x})$ is therefore a constant for these prices, the vector $\mathbf{x}' - \mathbf{x}$ is normal to F. Call the greatest common divisor of the entries of $\mathbf{x}' - \mathbf{x}$ the *weight* of the facet, $w(F)$. So $\mathbf{v}_F := \frac{1}{w(F)}(\mathbf{x}' - \mathbf{x})$ is a primitive integer vector (that is, the greatest common divisor of its entries is 1), and it points from the UDR where $\mathbf{x}'$ is demanded to the UDR where $\mathbf{x}$ is. But since F is $(n-1)$ dimensional, its normal direction is unique, so there is a unique primitive integer normal vector pointing from the UDR of $\mathbf{x}'$ to that of $\mathbf{x}$. Thus knowing only F, $w(F)$ and $\mathbf{x}$ allows us to derive $\mathbf{v}_F$, and hence $\mathbf{x}'$. It therefore follows that if we know demand in any one UDR, we can find demand everywhere from knowing the set of facets (and hence their primitive integer normal vectors) and their weights.

A rational polyhedral complex is described as *weighted* if a positive integer weight is attached to each facet. We provide examples in 2.4.

Understanding these weightings allows us to now give the full, formal definition of a tropical hypersurface – recall that we have so far worked only with the underlying set. For completeness we repeat the definition of that set here, and so:[23]

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[23]These definitions are mathematically identical to those of Mikhalkin (2004 and subsequent work), but the mathematical literature has not, of course, interpreted them in an economic context (that is, understood the $D_u(p)$ as demand sets).

Definition 2.3 (Mikhalkin, 2004, Example 2)**.** Let $A \subsetneq \mathbb{Z}^n$ be a finite set and let $u : A \to \mathbb{R}$ be any function. Then the *tropical hypersurface* $\mathcal{T}_u$ associated with u is the weighted rational polyhedral complex such that:

1. its underlying set is $\{\mathbf{p} \in \mathbb{R} \mid \#D_u(\mathbf{p}) > 1\}$;

2. the weight $w(F)$ of the facet F is the integer defined by $w(F)\mathbf{v}_F = \mathbf{x}' - \mathbf{x}$, in which $\mathbf{x}'$ is demanded in the UDR on one side of F, and $\mathbf{x}$ is demanded in the UDR on the other side, while $\mathbf{v}_F$ is the primitive integer normal vector pointing from the former to the latter.

We will see that the TH captures all the information we might ever need to know about an agent's demand and valuation function, if the latter is concave in the standard sense:

Definition 2.4. A function $u : A \to \mathbb{R}$ is *concave* if $(\text{Conv}\,A) \cap \mathbb{Z}^n = A$ and if u can be extended to a weakly concave function on $\mathbb{R}^n$.

It is a standard result that concave functions are precisely those for which there are no bundles in A that are never demanded (see, e.g., Milgrom and Strulovici, 2009, Theorem 1). That is:

Lemma 2.5. *Let $A \subset \mathbb{Z}^n$. A function $u : A \to \mathbb{R}$ is concave iff, for all $\mathbf{x} \in A$, there exists $\mathbf{p} \in \mathbb{R}^n$ such that $\mathbf{x} \in D_u(\mathbf{p})$.*

Note, however, that we do not assume that all valuations are concave.

It will be very important in our considerations of equilibrium (see Section 6) to know that, if bundles are demanded, they are demanded at the 'natural' price:

Lemma 2.6 (Pseudo-equilibrium Prices Lemma, Milgrom and Strulovici, 2009, Proposition 2)**.** *Let u be **any** valuation function. Suppose $\mathbf{p}$ is any price vector, and $\mathbf{x}$ is an integer bundle in $\text{Conv}\,D_u(\mathbf{p})$. If there exists any price vector $\mathbf{p}'$ such that $\mathbf{x} \in D_u(\mathbf{p}')$, then $\mathbf{x} \in D_u(\mathbf{p})$.*

Proof. For all $\mathbf{x}^\beta \in D_u(\mathbf{p})$, we know $u(\mathbf{x}) - \mathbf{p}.\mathbf{x} \le u(\mathbf{x}^\beta) - \mathbf{p}.\mathbf{x}^\beta$, with equality only if $\mathbf{x} \in D_u(\mathbf{p})$. So if $\mathbf{x} \in \text{Conv}\,D_u(\mathbf{p})$, i.e., $\mathbf{x} = \sum_\beta \lambda_\beta \mathbf{x}^\beta$ for some $\lambda_\beta \in [0,1]$ with $\sum_\beta \lambda_\beta = 1$, then it follows that $u(\mathbf{x}) - \mathbf{p}.\mathbf{x} = \sum_\beta \lambda_\beta (u(\mathbf{x}) - \mathbf{p}.\mathbf{x}) \le \sum_\beta \lambda_\beta u(\mathbf{x}^\beta) - \sum_\beta \lambda_\beta \mathbf{p}.\mathbf{x}^\beta = \sum_\beta \lambda_\beta u(\mathbf{x}^\beta) - \mathbf{p}.\mathbf{x}$ and so, simplifying, that $u(\mathbf{x}) \le \sum_\beta \lambda_\beta u(\mathbf{x}^\beta)$, with equality only if $\mathbf{x} \in D_u(\mathbf{p})$.

Now suppose $\mathbf{x} \in D_u(\mathbf{p}')$. Then $u(\mathbf{x}) - \mathbf{p}'.\mathbf{x} \ge u(\mathbf{x}^\beta) - \mathbf{p}'.\mathbf{x}^\beta$ for all $\mathbf{x}^\beta$ so we similarly show that $u(\mathbf{x}) \ge \sum_\beta \lambda_\beta u(\mathbf{x}^\beta)$. Hence, if $\mathbf{x} \in D_u(\mathbf{p}')$ for any $\mathbf{p}'$, then $\mathbf{x} \in D_u(\mathbf{p})$. $\qquad\square$

2.3   Associating demand with geometric objects

When does a weighted rational polyhedral complex depict a valid demand of some agent?

If we construct a TH by starting from some valuation function u, then the weights we attach will necessarily be coherent, in the sense that if we cross facets by passing through a sequence of UDRs that ends where it started, we must demand at the end precisely what we demanded at the beginning. In particular, the TH will satisfy the *balancing condition*:

Definition 2.7 (Mikhalkin, 2004, Definition 3). An $(n-1)$-dimensional weighted rational polyhedral complex $\Pi \subsetneq \mathbb{R}^n$ is *balanced* if for every for every $(n-2)$-cell $G \subsetneq \Pi$, the weights $w(F_j)$ on the facets $F_1, \ldots, F_l$ that are adjacent to G, and primitive integer normal vectors $\mathbf{v}_{F_j}$ for these facets that are defined by a rotational direction about G, satisfy $\sum_{j=1}^l w(F_j)\mathbf{v}_{F_j} = 0$.[24]

Note that there do not necessarily exist weights to balance a general rational polyhedral complex.[25] However, the balancing condition is in fact the *only* condition that a weighted rational polyhedral complex has to satisfy to be the TH of some valuation function:

Theorem 2.8 (Mikhalkin, 2004, Proposition 2.4; also Mikhalkin, 2005, Theorem 3.15). *Suppose that $\Pi \subsetneq \mathbb{R}^n$ is an $(n-1)$-dimensional balanced weighted rational polyhedral complex.[26] Then there exists a finite set $A \subsetneq \mathbb{Z}^n$ and a function $u : A \to \mathbb{R}$ such that Π is the TH, $\mathcal{T}_u$.*

The correspondence between a TH and its associated set A and function u is not unique, but the ambiguities are trivial if u is concave. Clearly, adding a constant to $u(\mathbf{x})$ leaves the TH unchanged, as does increasing every available bundle by a fixed bundle and making a corresponding shift in the valuation (though the bundle demanded in each UDR will then also be increased by the fixed bundle). That is, if $A' = \{\mathbf{x} + \mathbf{c} \mid \mathbf{x} \in A\}$ and $u'(\mathbf{x} + \mathbf{c}) = u(\mathbf{x}) + \alpha$ for all $\mathbf{x} \in A$, some $\mathbf{c} \in \mathbb{Z}^n$, and some $\alpha \in \mathbb{R}$, then $\mathcal{T}_{u'} = \mathcal{T}_u$. (See Example 2.12 for an example of such a shift).

Furthermore, any non-concave u has the same TH as the minimal weakly-concave function that weakly exceeds it everywhere on A. To see this, observe that if a bundle is never demanded then its precise value to the agent is immaterial, so we can increase its value up to the threshold at which it is just marginally demanded for some price(s) without altering the shape or properties of the TH. Doing this for all never-demanded bundles removes any non-concavities in the valuation function. It is also now clear that if two agents have valuations u and u', respectively on different sets of bundles A and A', but their convex hulls in $\mathbb{R}^n$, which we write Conv A and Conv A', coincide; and if $\widehat{u}$ is the minimal concave function on Conv A such that $\widehat{u} \geq u$ on A, and is also the minimal concave function on Conv A such that $\widehat{u} \geq u'$ on A'; then $\mathcal{T}_{\widehat{u}} = \mathcal{T}_u = \mathcal{T}_{u'}$.[27]

Summing up:

Theorem 2.9 (Mikhalkin, 2004, Remark 2.3). *There is a 1-1 correspondence between THs with an identified 'demand 0' UDR, and pairs (u, A), where $A \subsetneq \mathbb{Z}^n$ is finite and*

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[24]To choose a rotational direction around G, pick a 2-dimensional affine subspace H of $\mathbb{R}^n$ orthogonal to G, such that the intersection of each F_j with H is 1-dimensional. The intersection of H with the TH is then a collection of 1-cells meeting at the 0-cell which is $G \cap H$. An ordinary choice of rotational direction in this two-dimensional picture gives a rotational direction around G in $\mathbb{R}^n$.

[25]For example, in two dimensions, consider three 0-cells, each with three adjacent facets. If each pair of 0-cells has an adjacent facet in common, the six weights must satisfy six balancing conditions (that is, three equations in each of the two dimensions). But since the balancing conditions are trivially satisfied by setting all weights equal to zero, the conditions can only be satisfied by positive integer weights if the conditions are not linearly independent–which is non-generic.

[26]Strictly speaking, of course, Π is a subset of the space $\mathbb{R}^n$ *and* has weights. As before, we follow Mikhalkin and the
mathematical literature in our presentation.

[27]We defined $\widehat{u}$ on Conv $A \subsetneq \mathbb{R}^n$, but it still defines a TH if it is restricted to (Conv $A) \cap \mathbb{Z}^n$.

convex in $\mathbb{Z}^n$, u is a weakly-concave, function u on A, for which $u(0) = 0$ and 0 is demanded where specified.

Thus we have full equivalence between THs and weakly-concave valuation functions (such that $u(0) = 0$ and 0 is demanded in a specified UDR). Note, in particular, that a given set in $\mathbb{R}^n$ is the TH of some quasilinear demand if and only if it is a rational polyhedral complex and there exist weights for the facets such that it is balanced. Although we do *not* restrict attention to concave valuation functions–indeed Section 6.3 will ask when an aggregate valuation is concave–understanding of the concave case is important.

Similarly, we do *not* restrict attention to what is demanded in UDRs, but doing so is an important first step. Generically all prices are in a UDR so, as noted above Definition 2.3, given any TH and a specified 'zero demand' UDR we can easily work out what is demanded for a generic price. And it is particularly straightforward to relate properties of demand such as substitutes or complements to the geometry of the TH; see Section 4.

2.4 Demand examples

Example 2.10. Let $A = \{\mathbf{x} \in \mathbb{Z}_{\geq 0}^2 \mid x_1 + x_2 \leq 2\}$ and let $u : A \to \mathbb{R}$ be as follows (we arrange the terms in this "back-to-front" way to correspond to the fact that *smaller* quantities will appear *higher* in, and further *right* in, the TH; this convention will be particularly helpful later):

$x_1 = 2$	$x_1 = 1$	$x_1 = 0$	u
7	6	0	$x_2 = 0$
	9	4	$x_2 = 1$
		8	$x_2 = 2$

The TH associated with the agent's valuation, u, is shown in Figure 3, in which we have additionally marked in red the bundle demanded by this agent in each UDR. The facet between the UDRs in which $(0,0)$ and $(0,2)$ are demanded has weight 2. For $\mathbf{p}$ in this facet (that is, for $p_2 = 4$ and $p_1 > 6$) we have $D_u(\mathbf{p}) = \{(0,0),(0,1),(0,2)\}$; in particular the bundle $(0,1)$ *is* demanded for some price and the function *is* concave. An otherwise-identical valuation u' in which $u'(0,1) < 4$ would give rise to the *same* TH, but would not be concave; $(0,1)$ would not be demanded for any price.

It is easy to work out, from the TH, which bundle is demanded in each UDR, if one already knows what is demanded in any one UDR. If $x_1 = x_2 = 0$ in the top right UDR we can simply "walk around" the diagram, adding to x_1 (x_2) the weight of any facet crossed times the first (second) coordinate of the primitive integer facet normal. Thus starting from the top right UDR, crossing the vertical facet with normal $(1,0)$, that is, $\{\mathbf{p} \in \mathbb{R}^2 \mid p_1 = 6, p_2 > 4\}$, changes demand from $(0,0)$ to $(1,0)$; from there, crossing the facet with normal $(-1,2)$ changes demand to $(0,2)$, as may also be seen by crossing the weight-2 horizontal from $(0,0)$ downwards; and so on.

Example 2.11. It will be useful later to discuss very simple examples of substitutes and complements demands: if $A = \{0,1\}^2$, then $u^1 : A \to \mathbb{R}$ and $u^2 : A \to \mathbb{R}$ defined as
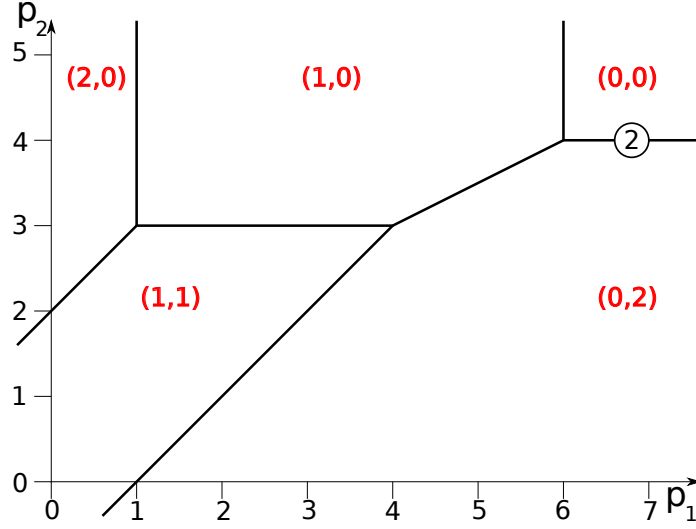
Figure 3: The TH of Example 2.10, with the bundle demanded in each UDR marked in red.

follows are demands for substitutes and complements, respectively, and their THs are shown in Figures 4a and 4b.[28]

$x_1 = 1$	$x_1 = 0$	u^1
1	0	$x_2 = 0$
1	1	$x_2 = 1$

and

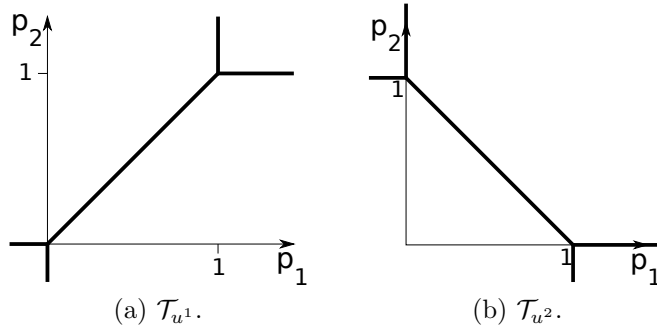| $x_1 = 1$	$x_1 = 0$	u^2
0	0	$x_2 = 0$
1	0	$x_2 = 1$

.

(a) $\mathcal{T}_{u^1}$.

(b) $\mathcal{T}_{u^2}$.

Figure 4: The THs of Example 2.11.

Clearly each TH has four UDRs in which these agents demand the bundles $(0,0)$, $(0,1)$, $(1,1)$, and $(1,0)$, respectively, as one moves clockwise around the UDRs starting at the top right–as is also easily confirmed by adding the appropriate primitive integer facet normal on every crossing between UDRs.

Example 2.12. To illustrate the case in which an agent both buys and sells goods, let $A = \{(0,0), (-1,1)\}$ and let $u(0,0) = 0$, and $u(-1,1) = -3$. This corresponds to an

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[28]The TH of Figure 4a appears to be a translation of Figure 1, but there is an important distinction. In Figure 1 the domain is $\{(0,0), (0,1), (1,0)\}$, so the TH has only one 0-cell; here, u^1 has domain $\{0,1\}^2$, and its TH has two 0-cells. (If we restricted u^1 to the domain $\{(0,0), (0,1), (1,0)\}$ its TH would coincide with $\mathcal{T}_{u^1}$ on $\mathbb{R}^2_{\geq 0}$ but have only one 0-cell.)
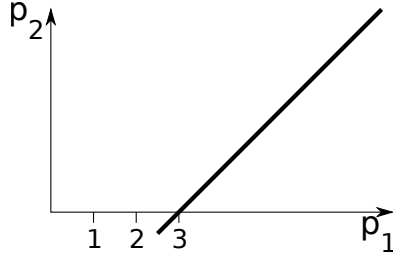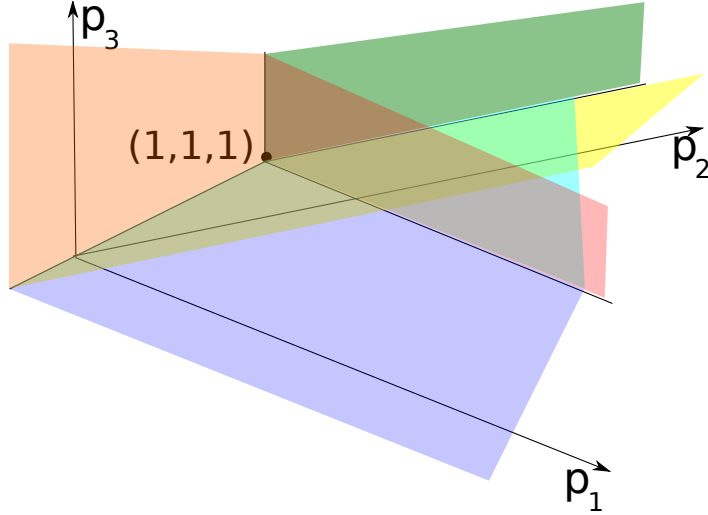
14

Figure 5: The TH of Example 2.12.

Figure 6: The TH of Example 2.13.

agent who can convert one unit of good 2 into one unit of good 1 at a cost of 3; the agent will therefore buy one unit of good 2 and sell one unit of good 1 if $p_1 - p_2 > 3$, and do nothing if $p_1 - p_2 < 3$. See Figure 5.

Observe that it would be economically identical if the agent were initially endowed with one unit of good 1 which it would be prepared to trade for a unit of good 2 if the price difference were at least 3–the agent's choices of what to buy and sell would depend on the prices in exactly the same way. This corresponds to simply shifting the valuation to the right so $A = \{(1,0),(0,1)\}$ with $u(1,0) = 0$ and $u(0,1) = -3$. Note that in this case $(0,0) \notin A$. We need not (and do not) prescribe that the zero bundle has to be an available option.

Example 2.13. For a simple 3-dimensional example, let $A = \{\mathbf{x} \in \mathbb{Z}_{\geq 0}^3 \mid x_1 + x_2 + x_3 \leq 1\}$ and let $u(0,0,0) = 0$ and $u(1,0,0) = u(0,1,0) = u(0,0,1) = 1$. The TH is given in Figure 6. Now, the facets are 2-dimensional (pieces of planes), there are additionally 1-cells (lines along which these facets meet), and a 0-cell (point at which these lines meet). Three of these facets, having normals $(1,0,0)$ (dark-green facet), $(0,1,0)$ (red facet), and $(0,0,1)$ (turquoise facet), border the UDR in which $(0,0,0)$ is demanded; this UDR is of course $\{\mathbf{p} \in \mathbb{R}^3 \mid p_1, p_2, p_3 > 1\}$. Crossing any one of these facets takes us to the UDR in which the corresponding bundle is demanded. We swap between the latter UDRs by crossing the remaining three facets, which have normals $(1,-1,0)$ (orange facet), $(0,1,-1)$ (bluish-purple facet) and $(1,0,-1)$ (yellow facet).

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2.5 Classic models interpreted in our framework

Many classic models in which agents have quasi-linear demands for indivisible goods are special cases of our framework:

First, it is trivial that Bikhchandani and Mamer (1997) is the restriction of our model to $A = \{0,1\}^n$.

Example 2.14 (Workers and Firms–Kelso and Crawford, 1982, and Hatfield and Milgrom, 2005). Kelso and Crawford (1982) model matching between workers, desiring at most one job, and firms, interested in hiring many workers, who they regard as substitutes. Thus each 'good' is a contract between a worker and a firm, and its 'price' is the salary.

To represent this in our framework, let $i \in \{1, \ldots, m_1\}$ be the workers, and $j \in \{1, \ldots, m_2\}$ be the firms, so there are $n = m_1 m_2$ contracts which we can index as (i, j). Then worker i has valuation $u^i : A^i \to \mathbb{R}$ with domain $A^i := \{\mathbf{0}, -\mathbf{e}^{(i,j)} \mid j = 1, \ldots, m_2\} \subsetneq \{-1, 0\}^n$. That is, we regard it as a seller of its labour, and it has no preferences over the sale of other workers' labour. On the other hand, firm j has valuation $u^j : A^j \to \mathbb{R}$ with domain $A^j := \{\mathbf{x} \in \{0,1\}^n \mid x_{(i,j')} = 0 \text{ for } j' \neq j\} \subsetneq \{0,1\}^n$. That is, it is only able to buy workers, and only has preferences over the workers it 'buys' itself. Note that the total set of meaningful bundles $\{-1, 0, 1\}^n$ is the (Minkowski) sum of all the sets A^i and A^j; in Section 6 we will refer to this set as the domain of the aggregate valuation. We discuss Kelso and Crawford's 'gross substitutes' condition in Section 5.3.

Hatfield and Milgrom (2005) consider matchings between firms and workers with more general 'contracts' than just salaries, but their model can be embedded in Kelso and Crawford (1982),[29] so can also be presented in our framework.

Example 2.15 (General 'Trading Networks'–Hatfield et al., 2013, Ostrovsky, 2008). Models such as Hatfield et al. (2013) consider agents each of whom can both buy and sell. Each 'good' in these models is the trade of a single unit of a product between a specified buyer and a specified seller; additional units of the identical product are treated as separate trades and may have differing prices.[30]

To embed this in our framework, let n be the total number of feasible trades, and for $j = 1, \ldots, n$ let $b(j)$ be the buyer and $s(j)$ be the seller in the (potential) trade. Then agent i has valuation $u^i : A^i \to \mathbb{R}$ with domain $A^i \subseteq \{\mathbf{x} \in \{-1, 0, 1\}^n \mid x_j < 0 \Rightarrow s(j) = i; \ x_j > 0 \Rightarrow b(j) = i\}$. That is, agent i considers bundles in which the goods it sells are in non-positive quantities, and the goods it buys are in non-negative quantities. However, agent i need not consider the *whole* of this set; there may be bundles that are infeasible (for example, if it cannot sell good 1 unless it also buys one of goods 2, 3 or 4, then bundle $-\mathbf{e}^1$ is not in the domain A^i). Example 2.12 is a special case of this model.

We discuss Hatfield et al. (2013)'s 'full substitutability' condition in Section 5.3.

Example 2.16 (Coalition Formation with Transferable Utility). [31] A classic literature

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[29]Echenique (2012) shows this. Hatfield and Kojima (2010) does not fit into our framework, since it is inconsistent with quasi-linear preferences (see the discussion in Echenique, 2012).

[30]Hatfield et al. (2013) impose no restrictions on the shape of the 'trading network' formed by the feasible trades, so thus generalise the 'two-sided matching literature' started by Gale and Shapley (1962) in the case in which all preferences are quasilinear. (Ostrovsky, 2008 is also related, but does not require quasi-linearity and has discrete prices.)

[31]Cf. Koopmans and Beckmann (1957), Shapley and Shubik (1971), Kaneko and Wooders (1982), Eriksson and Karlander (2001), Talman and Yang (2011) and Chiappori et al. (2012).

models coalition formation – typically in a bipartite context. Each person (we do not refer to people as 'agents', since they will not take the roles of 'agents' in our framework) gets intrinsic value from being a member of a coalition. However, the surplus of any coalition can be transferred among people within that coalition, in the form of side-payments. Each person has quasi-linear utility in the intrinsic value of the coalition and this side-payment. We typically seek a 'stable' outcome, in which every person is assigned to precisely one coalition (perhaps entailing them being alone) and no subset of people can all (strictly) gain by deviating from their prescribed coalition and forming a new one.[32]

We model this in our framework as follows: the 'agents' are potential *coalitions*, who buy 'goods', which are *people*. The 'price' paid by an 'agent' for a good is the surplus (including any side payments) the person receives in that coalition.

So the goods available are 'person-goods' indexed $i = 1, \ldots, n$. The bundles $\mathbf{x} \in \{0,1\}^n$ denote sets of 'person-goods', where $x_i = 1$ iff i is included in the set. We let B be the set of feasible coalitions; in general, not every set of people is a feasible coalition, but we include a distinct 'coalition-agent' for every coalition that is feasible, including any feasible coalition that yields zero utility (for example, a given person being alone may be a feasible coalition that yields zero utility).

We consider the people as each being assigned to a coalition and immediately handing their value in that coalition to the coalition-agent itself. Some of this money is then transferred back to the people in the coalition: p_i is the *price* the coalition-agent pays for person-good i. If a person-good is priced at p_i then the *net* side-payment to this person from coalition-agent $\mathbf{x}$ is thus $p_i - u^i(\mathbf{x})$, where $u^i : \{\mathbf{x} \in B \mid x_i = 1\} \to \mathbb{R}$ is the individual's intrinsic valuation function on coalitions. Hence, the *net utility* to person i at this stage is simply $u^i(\mathbf{x}) + p_i - u^i(\mathbf{x}) = p_i$. (There will in general be additional surplus to distribute among the coalition members at a later stage; we think of the 'price' as the minimum that needs to be offered to buy the person-good.)

Thus we think of each person as stating a price for himself[33] and seeing which coalition will buy; although his intrinsic values for the different coalitions may differ, his net utilities when he receives this price are all the same.

The 'coalition-agent' corresponding to coalition $\mathbf{x}$ obtains the sum of the individual values of the people in that coalition *minus* the 'prices' it pays for those people, if they are all assigned to it. So the domain of the coalition-agent's valuation is $A^{\mathbf{x}} := \{\mathbf{0}, \mathbf{x}\}$ and we define $u^{\mathbf{x}}(\mathbf{0}) = 0$ and $u^{\mathbf{x}}(\mathbf{x}) = \sum_{i \,:\, x_i = 1} u^i(\mathbf{x})$. If the vector of prices for person-goods is $\mathbf{p}$, the coalition-agent's net utility from bundle $\mathbf{y}$ is $u^{\mathbf{x}}(\mathbf{y}) - \mathbf{p}.\mathbf{y}$. So if the sum of the 'prices' of all the people is at most the total surplus from the coalition, that is, the coalition-agent can make side-payments that give each person the utility he demands, then the coalition-agent's maximising bundle is $\mathbf{x}$; otherwise it is $\mathbf{0}$. Thus the formation of coalitions is just the maximising behaviour of coalition-agents.

We discuss the formation of stable coalitions in equilibrium in Section 6.2.

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[32]We will see (in Section 6.2) that in this setting the stable outcomes will coincide with the core allocations (and also coincide with the core allocations of a game with fully transferable utility, i.e., across as well as within coalitions).

[33]We prefer the use of the female pronoun for people, except where–as here–they are treated as goods to be priced and traded.

3 Duality in Tropical Geometry

The previous section demonstrated the equivalence between THs and specific valuation functions. However, we now describe a coarser correspondence between a *set* of THs that are "essentially" the same as one another, and *sets* of valuation functions which–we will see–have the same fundamental properties.[34]

Looking, e.g., at Figure 1, the important structure is that there are particular UDRs and particular sets of prices at which the agent is indifferent between the bundles of those UDRs. So we say that two THs have the same *combinatorial type* if there is a 1-1 correspondence between the cells of the THs which have the same dimension and slope, and these cells connect to one another in the same way. Demands corresponding to THs of the same combinatorial type are "essentially" the same in that they represent agents who make the same trade-offs between additional units of goods, even if not always at the same prices. We will show that all THs of the same combinatorial type are, in a precise way, dual to a particular subdivision of Conv A.

3.1 Duality between convex polytopes and cells

Although we assume that goods are indivisible, we now develop a structure of convex polytopes and their faces in quantity space, so extend our focus from A to Conv $A \subsetneq \mathbb{R}^n$. We first show that this extension has no effect on the way we separate prices into different cell interiors by showing that Conv $D_u(\mathbf{p}) = $ Conv $D_u(\mathbf{p}') \iff D_u(\mathbf{p}) = D_u(\mathbf{p}')$, for any prices $\mathbf{p}$ and $\mathbf{p}'$. This is an immediate corollary of Lemma 2.6.

Corollary 3.1. *For **any** valuation function, u, if $\mathbf{p}$ and $\mathbf{p}'$ are any two price vectors, then* Conv $D_u(\mathbf{p}) = $ Conv $D_u(\mathbf{p}') \iff D_u(\mathbf{p}) = D_u(\mathbf{p}')$.

Proof. It is immediate from Lemma 2.6 that if $\mathbf{x} \in D_u(\mathbf{p}') \subseteq $ Conv $D_u(\mathbf{p}') = $ Conv $D_u(\mathbf{p})$ then $\mathbf{x} \in D_u(\mathbf{p})$, so the result follows. $\square$

For any price, $\mathbf{p}$, we write $\Delta(\mathbf{p}) := $ Conv $D_u(\mathbf{p})$ for this polytope in (divisible) quantity space $\mathbb{R}^n$. From Definition 2.1, and Corollary 3.1 we can write the associated cell interior as $\{\mathbf{p}'' \in \mathbb{R}^n \mid \Delta(\mathbf{p}) = \Delta(\mathbf{p}'')\}$, and since it is therefore defined by the polytope $\Delta(\mathbf{p})$, we write $C_{\Delta(\mathbf{p})}$ for the corresponding cell (its closure). Recall from the discussion in Section 2.2 that a price $\mathbf{p}''$ is in the cell $C_{\Delta(\mathbf{p})}$ iff the bundles in $D_u(\mathbf{p})$ are weakly preferred to all others at price $\mathbf{p}''$, i.e., iff $D_u(\mathbf{p}) \subseteq D_u(\mathbf{p}'')$.[35] Applying Corollary 3.1 again, we conclude that $C_{\Delta(\mathbf{p})} = \{\mathbf{p}'' \in \mathbb{R}^n \mid \Delta(\mathbf{p}) \subseteq \Delta(\mathbf{p}'')\}$. It follows immediately

$$\Delta(\mathbf{p}) \subsetneq \Delta(\mathbf{p}') \iff C_{\Delta(\mathbf{p}')} \subsetneq C_{\Delta(\mathbf{p})}. \tag{2}$$

We now describe the dualities between the polytope $\Delta(\mathbf{p})$ in quantity space, and the associated cell $C_\Delta(\mathbf{p})$ in price space; we show how they extend to the global structure in the next subsection.

First, note the dimensions of $\Delta(\mathbf{p})$ and $C_{\Delta(\mathbf{p})}$ are dual. $C_{\Delta(\mathbf{p})}$ has the dimension of its affine span, that is, of that set of prices $\mathbf{p}'$ such that $\mathbf{p}'.(\mathbf{x} - \mathbf{x}') = u(\mathbf{x}) - u(\mathbf{x}')$ for all

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[34]For much more on this Legendre-Young duality, see Murota (2003, especially Chapter 8).
[35]See also Footnote 22.

$\mathbf{x}, \mathbf{x}' \in D_u(\mathbf{p})$. If $\Delta(\mathbf{p})$ is k-dimensional, these equations impose k linearly independent constraints on such $\mathbf{p}'$, so $\dim C_{\Delta(\mathbf{p})} = n - k$.

Next observe the affine spans of these sets are orthogonal: since $\mathbf{p}'.(\mathbf{x}-\mathbf{x}')$ is constant for all $\mathbf{p}' \in C_{\Delta(\mathbf{p})}$ and all $\mathbf{x}, \mathbf{x}' \in D_u(\mathbf{p})$, we have $(\mathbf{p}' - \mathbf{p}'').(\mathbf{x} - \mathbf{x}') = 0$ for any $\mathbf{p}', \mathbf{p}'' \in C_{\Delta(\mathbf{p})}$ and $\mathbf{x}, \mathbf{x}' \in \Delta(\mathbf{p})$. So all prices in $C_{\Delta(\mathbf{p})}$ lie in a subspace of $\mathbb{R}^n$ orthogonal to any $\mathbf{x} - \mathbf{x}'$ where $\mathbf{x}, \mathbf{x}' \in \Delta(\mathbf{p})$, and all bundles in $\Delta(\mathbf{p})$ lie in a subspace of $\mathbb{R}^n$ orthogonal to $\mathbf{p}' - \mathbf{p}''$ for any $\mathbf{p}', \mathbf{p}'' \in C_{\Delta(\mathbf{p})}$.

Therefore, any ($n-1$ dimensional) facet $F = C_{\Delta(\mathbf{p})}$ (in price space) corresponds to a 1-dimensional polytope, i.e., a line-segment, $\Delta(\mathbf{p})$, orthogonal to it (in quantity space). And if $\mathbf{x}$ and $\mathbf{x}'$ are the endpoints of the line-segment $\Delta(\mathbf{p})$, then $\mathbf{x} - \mathbf{x}' = w\mathbf{v}_F$ for some $w \in \mathbb{Z}$, where $\mathbf{v}_F$ is a primitive integer vector in the direction of $\Delta(p)$, i.e. in the normal direction to F; let us chose $\mathbf{v}_F$ so that $w > 0$. And since the bundles demanded in the UDRs on either side of F are precisely the vertices at the endpoints of $\Delta(\mathbf{p})$, it also follows that this w is the weight of F, as defined in Section 2.2. In words, the "length" of the line-segment $\Delta(\mathbf{p})$ in quantity space is the weight of its corresponding facet in price space.

3.2 The subdivided Newton Polytope

Convex geometry now provides a clever trick to find the set of all the polytopes, $\Delta(\mathbf{p})$, very quickly, and see how they fit together in quantity space. From this it is easy to see how the cells of the TH fit together in price space.

The condition that a bundle $\mathbf{x}' \in D_u(\mathbf{p})$ maximises the agent's surplus at price $\mathbf{p}$ can be re-written using vectors in $\mathbb{R}^{n+1}$ as $(-\mathbf{p}, 1).(\mathbf{x}, u(\mathbf{x})) \leq (-\mathbf{p}, 1).(\mathbf{x}', u(\mathbf{x}'))$ for all $\mathbf{x} \in A$. So the points $(\mathbf{x}, u(\mathbf{x}))$, for all $\mathbf{x} \in A$, must lie in a particular half-space of $\mathbb{R}^{n+1}$. Furthermore all the other bundles $\mathbf{x}''$ which are optimal at the same price $\mathbf{p}$ satisfy $(-\mathbf{p}, 1).(\mathbf{x}'', u(\mathbf{x}'')) = (-\mathbf{p}, 1).(\mathbf{x}', u(\mathbf{x}'))$ and so all lie on the hyperplane in $\mathbb{R}^{n+1}$ bounding this half-space. Hence every set $\Delta(\mathbf{p})$ (i.e. any $\mathrm{Conv}\, D_u(\mathbf{p})$) is the projection to the first n coordinates of a face of the set

$$\widehat{A} := \mathrm{Conv}\{(\mathbf{x}, u(\mathbf{x})) \in \mathbb{R}^{n+1} \mid \mathbf{x} \in A\}. \tag{3}$$

Conversely, consider any face $\widehat{\Delta}$ of $\widehat{A}$ on the 'upper side' with respect to the final coordinate (i.e., any face such that points with a slightly lower final coordinate than those in the face are in $\widehat{A}$, and those with a slightly higher final coordinate are not). $\widehat{\Delta}$ is the intersection of $\widehat{A}$ with some hyperplane $\{\mathbf{y} \in \mathbb{R}^{n+1} \mid \mathbf{v}.\mathbf{y} = \alpha\}$ for some $\alpha \in \mathbb{R}$, and some normal vector $\mathbf{v} \in \mathbb{R}^{n+1}$. We know $\widehat{A}$ is contained in the half-space below the hyperplane with respect to the final coordinate. Renormalising so the final coordinate of $\mathbf{v}$ is 1, so $\mathbf{v} = (-\mathbf{p}, 1)$ for some vector $\mathbf{p} \in \mathbb{R}^n$, the face $\widehat{\Delta}$ is the convex hull of all points $(\mathbf{x}', u(\mathbf{x}'))$, where $\mathbf{x}'$ is in A, satisfying $(-\mathbf{p}, 1).(\mathbf{x}, u(\mathbf{x})) \leq (-\mathbf{p}, 1).(\mathbf{x}', u(\mathbf{x}'))$ for all $\mathbf{x} \in A$; that is, $u(\mathbf{x}') - \mathbf{p}.\mathbf{x}'$ is maximal over bundles in A. Thus the projection of $\widehat{\Delta}$ to its first n coordinates is exactly $\Delta(\mathbf{p})$ for this $\mathbf{p}$.

Summarising, each upper face of $\widehat{A}$ is the set $(\mathbf{x}, u(\mathbf{x}))$ that are maximal when viewed in the direction of some vector $(-\mathbf{p}, 1)$; the face then projects to $\Delta(\mathbf{p})$. And conversely, any set $\Delta(\mathbf{p})$ is the projection of an upper face of $\widehat{A}$. So the information about the demand sets is contained in the projections of these faces, that is, in the collection of

sets $\big\{\mathbf{x} \mid (\mathbf{x}, u(\mathbf{x})) \in \widehat{\Delta}\big\}$, where $\widehat{\Delta}$ is an upper face of $\widehat{A}$.

Definition 3.2.

1. The subdivision of Conv A given by the projections of the upper faces of $\widehat{A}$ onto Conv A is a *subdivided Newton polytope* (SNP).[36]

2. The image Δ of a k-dimensional face $\widehat{\Delta}$ of $\widehat{A}$ is a *k-face* of the SNP.

We give an example of how to construct an SNP in practice in Section 3.3.

Since, for $k < n$, any k-face of $\widehat{A}$ is the face of an n-face of $\widehat{A}$, it is sufficient to consider only the maximal faces of $\widehat{A}$ to identify the full SNP structure.

In particular, an SNP n-face, Δ, is the projection of an upper n-face $\widehat{\Delta}$ of $\widehat{A}$. But since $\widehat{\Delta}$ is n-dimensional, there is a unique hyperplane of $\mathbb{R}^{n+1}$ passing through it, and so a unique normal vector of the form $(-\mathbf{p}, 1)$. So the projection Δ of $\widehat{\Delta}$ to Conv A is exactly $\Delta(\mathbf{p}) = \text{Conv}\, D_U(\mathbf{p})$, and is *not* $\Delta(\mathbf{p}')$ for any $\mathbf{p}' \neq \mathbf{p}$. So $\mathbf{p}$ is the *only* price at which *all* these bundles are demanded, and $\{\mathbf{p}\}$ is therefore a 0-cell in the TH, i.e. $\{\mathbf{p}\} = C_{\Delta(\mathbf{p})}$.

At the other extreme, for any 0-face $\{\mathbf{x}\}$ of the SNP, there exist prices $\mathbf{p}$ at which $(\mathbf{x}, u(\mathbf{x}))$ is the unique point of $\widehat{A}$ intersecting a supporting hyperplane normal to $(-\mathbf{p}, 1)$. For any such $\mathbf{p}$ we know $\{\mathbf{x}\} = D_u(\mathbf{p})$. Furthermore, if any such $\mathbf{p}$ is changed infinitesimally in any coordinate direction, the point $\{(\mathbf{x}, u(\mathbf{x}))\}$ is still the unique point of $\widehat{A}$ intersecting the corresponding supporting hyperplane. So the UDR in which $\mathbf{x}$ is demanded, that is, the set of $\mathbf{p}$ such that $\{\mathbf{x}\} = D_u(\mathbf{p})$, is (of course) n-dimensional.

Between these extremes, any upper k-face of $\widehat{A}$, where $2 \leq k \leq n - 1$, is the intersection of $\widehat{A}$ with any one of a range of hyperplanes in $\mathbb{R}^{n+1}$. The vector $(-\mathbf{p}, 1)$ normal to any such hyperplane defines a price $\mathbf{p}$ lying in the corresponding $(n - k)$-dimensional cell interior of the TH.

Note also that, since Conv A need *not* in general be n-dimensional (see Example 2.12 for an example in which it is not) the SNP need not have any n-faces; this corresponds to a TH with no 0-cells (such as that in Figure 5).

The fact that the SNP's faces, $\Delta(\mathbf{p})$, are the projections of faces of a convex set tells us how they fit together, and hence how the sets $D_u(\mathbf{p})$ fit together. If $\Delta(\mathbf{p}) \subsetneq \Delta(\mathbf{p}')$ for two faces of the SNP, then $\Delta(\mathbf{p})$ must be a face of the polytope $\Delta(\mathbf{p}')$. But recall (displayed equation 2) that $\Delta(\mathbf{p}) \subsetneq \Delta(\mathbf{p}')$ iff $C_{\Delta(\mathbf{p}')} \subsetneq C_{\Delta(\mathbf{p})}$. As discussed above (at and beneath point **V** of Section 2.2,) the latter holds iff $C_{\Delta(\mathbf{p}')}$ is in the boundary of $C_{\Delta(\mathbf{p})}$. Moreover, $\Delta(\mathbf{p})$ and $C_{\Delta(\mathbf{p})}$ are orthogonal, as discussed in Section 3.1. So knowing how the $\Delta(\mathbf{p})$ fit together in quantity space makes it immediately obvious how the $C_{\Delta(\mathbf{p})}$ fit together in price space, and vice versa.

So the SNP tells us which cells must exist in the corresponding THs, their slopes, and how they are connected. In other words

Theorem 3.3 (Mikhalkin, 2004, Proposition 2.1.)**.** *For a given* Conv A *there is a 1-1 correspondence between SNPs of THs and combinatorial types of THs.*

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[36]It is a subdivision of the set Conv A which is itself called a Newton polytope in (tropical) algebraic geometry.

As noted above, this correspondence is coarser than the correspondence we described in the previous subsection (Theorem 2.9): different valuations correspond to the same SNP, and hence to a TH of same combinatorial type, even though the coordinates of the parts of the TH differ. However, this correspondence isolates the underlying properties of demands, specifically the sets of bundles one might ever be indifferent between, and the trade-offs one might make.

Also, starting from any SNP, it is easy to find the combinatorial type of the TH, and so see which coordinates uniquely define the TH. The TH can then be completely identified using the valuation u. We illustrate this in Section 3.3.

Another important point follows: if A is small, it is easy to list all the possible SNPs, and hence also list all possible combinatorial types of THs for the set A. That is, we can easily list every possible distinct structure of trade-offs that an agent might make between a given finite collection of goods.

Of course, we do not need to start with the SNP. Given the TH and an identified 'demand 0' UDR, we can easily infer both A and the full SNP using the duality described in this section.

Note, however, that if we do not know *ex ante* whether a TH is concave, then neither the TH nor the SNP can necessarily tell us which bundles are demanded in each cell of the TH. The information we do have is as follows:

Corollary 3.4. *Let A be convex in $\mathbb{Z}^n$, let $u : A \to \mathbb{R}$ be a valuation, and consider the corresponding SNP.*

1. *A bundle $\mathbf{x} \in A$ is a vertex of the SNP iff it is demanded in some UDR of the corresponding TH.*

2. *If every bundle $\mathbf{x} \in A$ is a vertex of the SNP, then $\widehat{u}$ is concave for every valuation $\widehat{u} : A \to \mathbb{R}$ such that $\mathcal{T}_{\widehat{u}} = \mathcal{T}_u$.*

3. *If a bundle $\mathbf{x} \in A$ is not a vertex of the SNP, there exist valuations $\widehat{u} : A \to \mathbb{R}$ such that $\mathcal{T}_{\widehat{u}} = \mathcal{T}_u$ but $\mathbf{x} \notin D_{\widehat{u}}(\mathbf{p})$ for any $\mathbf{p} \in \mathbb{R}$.*

Proof. 1 is clear from the previous discussion. 2 follows from Lemma 2.5. For 3, define $\widehat{u}$ to be equal to u on the vertices of the SNP, and to be arbitrarily large negative numbers on those bundles in A that are not vertices of the SNP. □

However, in quantity space we do *not* have an analogue of Theorem 2.8. Nor does there seem to be any simple analogue of Theorem 2.8's easy balancing condition that would check whether a given subdivision is the SNP of a valuation function:

Fact 3.5. It is *not* the case that every subdivision of every Newton polytope is the SNP of some valuation function.

Proof. A counterexample is provided by Gathmann (2006, Figure 7). □

3.3 Examples

Example 3.6. Starting from a valuation function, a TH can easily be drawn by first deriving the SNP, then using the SNP to draw the shape of the TH's combinatorial type, and finally using the valuations to fix the TH's exact location in price space.

Figure 7 presents a valuation function u, both in the usual tabular representation, and by showing the permissible set of bundles A, as a subset of the lattice $\mathbb{Z}^n$, labelled with their valuations. As before, the quantity of good 1 increases as we move to the *left*, and the quantity of good 2 increases as we move *down*, in order to show the duality between the SNP and the TH most clearly.

$x_1 = 2$	$x_1 = 1$	$x_1 = 0$	u
10	5	0	$x_2 = 0$
12	8	7	$x_2 = 1$
13	13	9	$x_2 = 2$

(a) Tabular representation of the valuation.

(b) Each circled number gives the value of the bundle in that position.
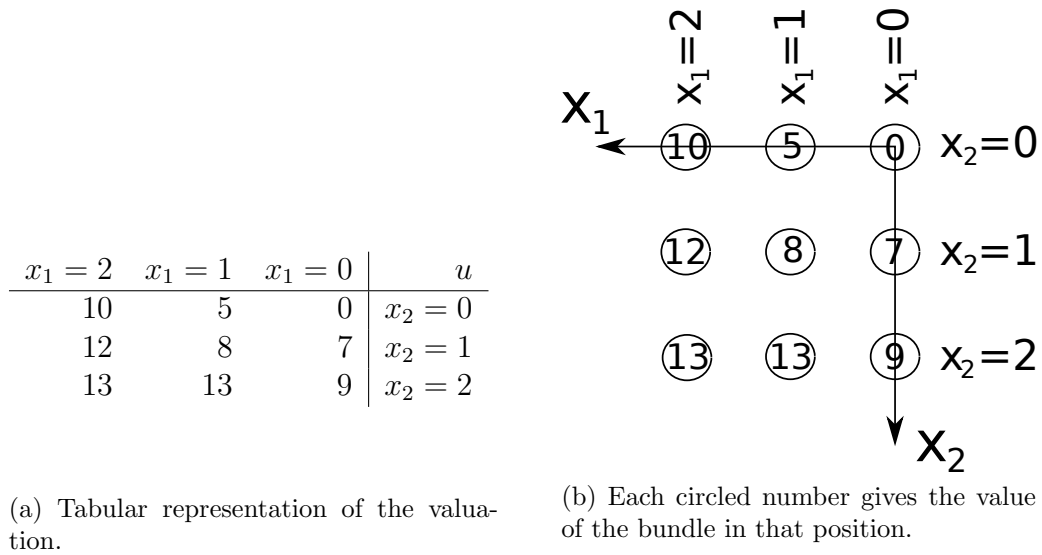
Figure 7: Alternative representations of a valuation function.

Figure 8 adds a third dimension to Figure 7b. Figure 8a shows the points $(\mathbf{x}, u(\mathbf{x}))$
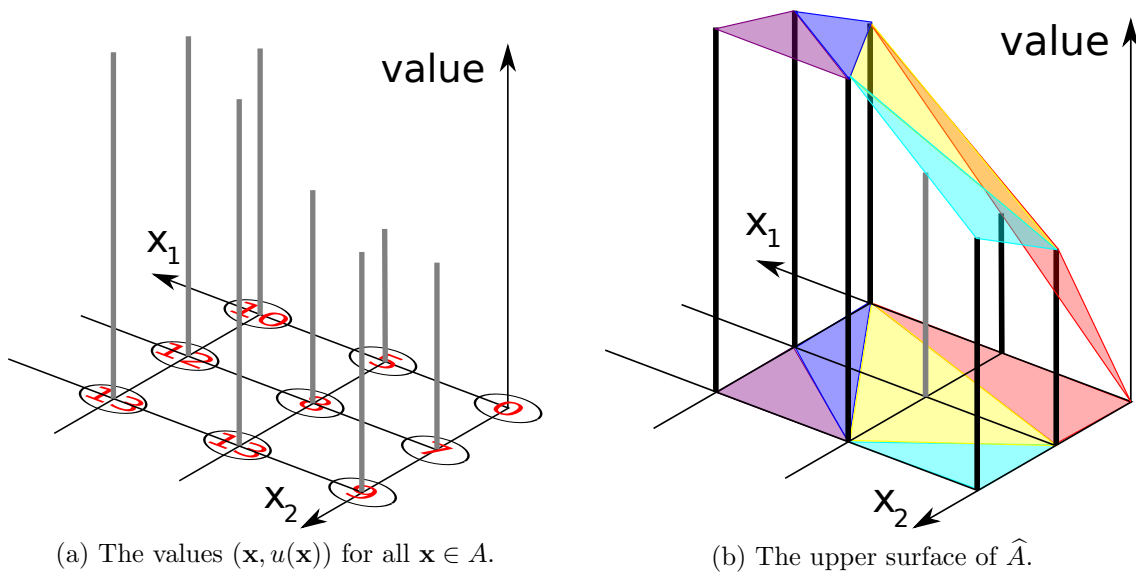

(a) The values $(\mathbf{x}, u(\mathbf{x}))$ for all $\mathbf{x} \in A$.

(b) The upper surface of $\widehat{A}$.

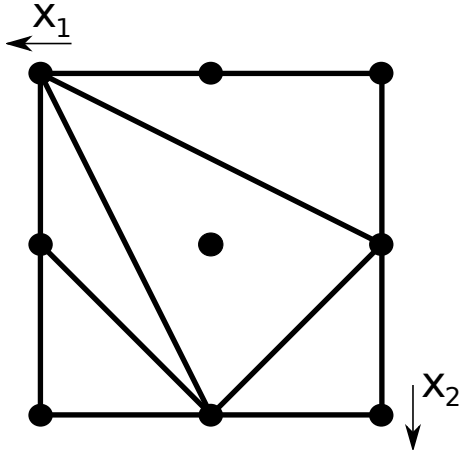Figure 8: Finding $\widehat{A}$ in three dimensions.

Figure 9: The SNP.

for all $\mathbf{x} \in A$, with the valuations $u(\mathbf{x})$ drawn as lines connecting them to their associated bundles, $\mathbf{x}$, to make the relationships clearer. Figure 8b then pictures the upper surface of $\widehat{A}$, with those lines that correspond to bundles that are demanded for some price(s) in bold. Note that the valuation is non-concave and the bundle $(1, 1)$ is never demanded.

The SNP is pictured in Figure 9. It is drawn *without* axes, since replacing A with $A + \mathbf{c}$ for some $\mathbf{c} \in \mathbb{Z}^n$ and re-defining u to correspond gives us the *same* SNP and TH. A depiction of the SNP and an example of a TH of the corresponding combinatorial type is given in Figure 10, colour-coded so that objects that are the geometric duals of

(a) The SNP, colour-coded.　　(b) A TH, colour-coded to correspond.
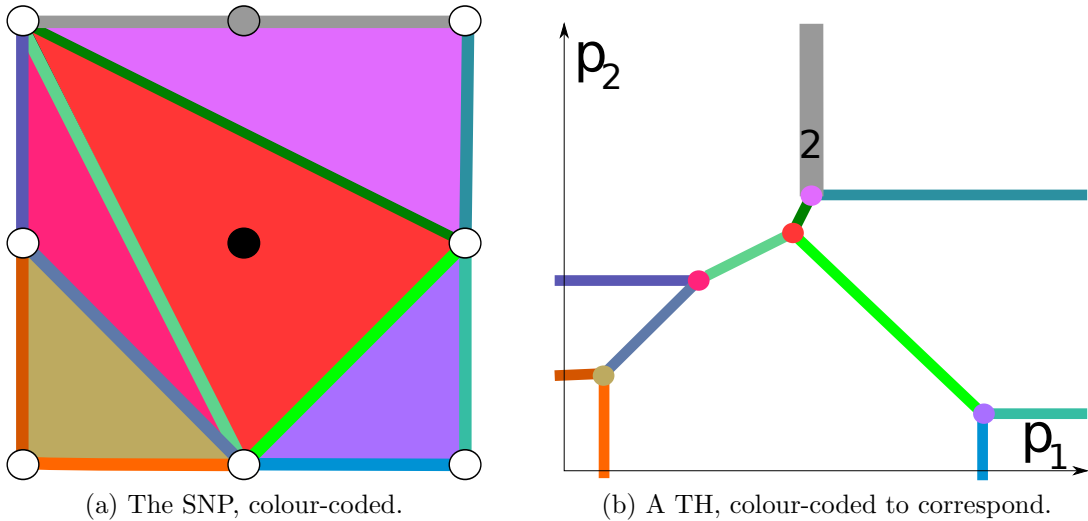
Figure 10: The SNP and a TH of the corresponding combinatorial type, colour-coded so that dual geometric objects have the same colours.

each other have the same colours as each other. That is, each point in the TH has the same colour as its corresponding area in the SNP; each line-segment (facet) in the TH has the same colour as the line-segment (edge) in the SNP that it corresponds, and is *orthogonal* to; and the white areas (UDRs) in the TH correspond to the white points (bundles that are vertices) in the SNP.

Note that the black point in the SNP that represents the bundle $(1, 1)$ has no object

Figure 11: The TH of the valuation function presented in Figure 7.

corresponding to it in the TH–it is "hidden" inside the scarlet-coloured point in the TH. If that bundle's valuation were greater so that, rather than the line corresponding to it in Figure 8b lying strictly below a plane coincident with $\widehat{A}$, the line instead just touched that plane,[37] then the bundle would be demanded at the price corres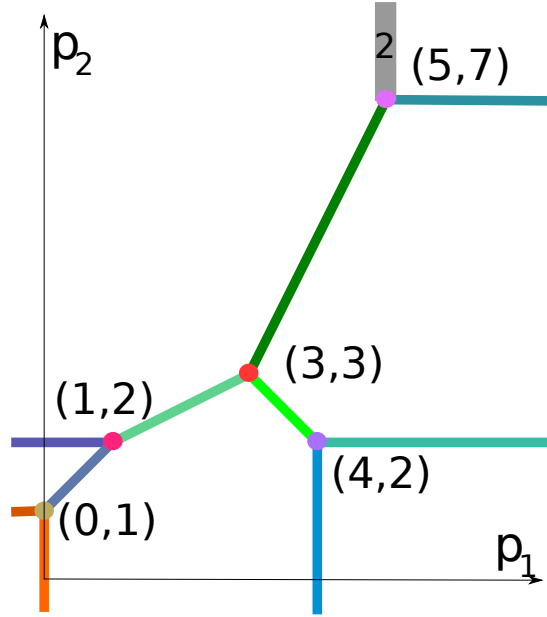ponding to the scarlet-coloured point in the TH. And if the bundle had a still higher valuation, that point in the TH would "open up" to form an area corresponding to the range of prices at which the bundle would then be demanded.

The final SNP lattice point is coloured grey. It is not an SNP vertex, but lies within (horizontal) SNP edge of the same colour, which has "length" 2 (more precisely, the greatest common divisor of the differences (2, and 0) between the co-ordinates of the bundles at the ends of this edge is 2). And this corresponds to the vertical grey facet in the TH which is labelled "2", reflecting its weight.

Finally, remember that Figure 10b shows only one of many THs of the combinatorial type corresponding to the SNP in that figure; the SNP is silent on the lengths of the lines in its corresponding THs. However, the exact location of the TH for our specific set of valuations can easily be worked out from the valuations of different bundles: See Figure 11.

For example, it is clear from the valuations of bundles (1,0) and (0,1) that the top right (pinky-purple) point of the TH is at $\mathbf{p} = (5, 7)$, since 5 and 7 are the prices below which the agent will first buy any of goods 1 and 2, respectively,when the other good's price is very high. And the coordinates of the purple point at the bottom right of the TH must be (4,2) since $9 - 7 = 2$ is the incremental value of a second unit of good 2, when the agent has no unit of good 1, and $13 - 9 = 4$ would be the further increment in value from then also having a unit of good 1, etc.

We discussed above (Section 2.2; see especially Example 2.10) how the demand in each UDR can easily be worked out from the TH.

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[37]It is easy to compute that the valuation of this bundle would have to be 10 for this to happen.

Example 2.10 revisited. It is not hard to check that the SNP for Example 2.10 is as shown in Figure 12a. Two examples of THs of the corresponding combinatorial type are given in Figures 12b and 12c.
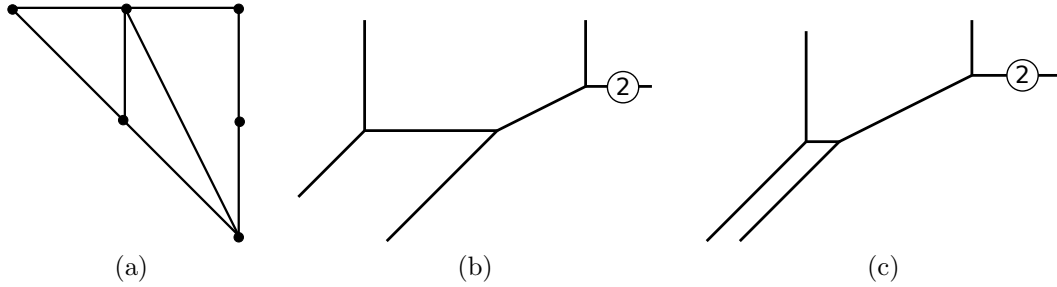

<div align="center">(a)   (b)   (c)</div>

Figure 12: (a) the SNP of Example 2.10; (b) and (c) two examples of THs of the combinatorial type of Example 2.10.

Example 3.7. For a fixed A, it is easy to draw every possible SNP and so obtain every possible combinatorial type of TH, thus enumerating all possible "essentially-different" structures of demand. We do this for $A = \{0,1\}^2$ in Figure 13.

It is not hard to see that Figure 13a applies when $u(0,0) + u(1,1) < u(1,0) + u(0,1)$, so represents substitutes; Figure 13b applies when $u(0,0) + u(1,1) = u(1,0) + u(0,1)$, so is additively separable demand; and Figure 13c applies when $u(0,0) + u(1,1) > u(1,0) + u(0,1)$, so is complements. (Recall Figure 4.) Importantly, it is clear that these are the *only* possibilities.

Observe that Figure 13b can be seen as a limit of Figure 13a (or, equivalently, Figure 13c). In the TH, the two 0-cells become arbitrarily close and then coincide in the limit; in the SNP, the faces of $\widehat{A}$ tilt until they are coplanar when the SNP edge distinguishing them disappears in this limit.

Likewise, any SNP in which the subdivision is not maximal (that is, additional valid $(n-1)$-faces could be added) can be recovered by deleting $(n-1)$-faces from some SNP whose subdivision *is* maximal; the corresponding TH is a limit (or 'degeneration'). Even for larger domains than $A = \{0,1\}^2$, we can efficiently enumerate all those combinatorial types of demand for which the SNP subdivision is maximal, knowing we can recover the remainder as their limits. We do this for $A = \{0,1,2\} \times \{0,1\}$ in Figure 14.
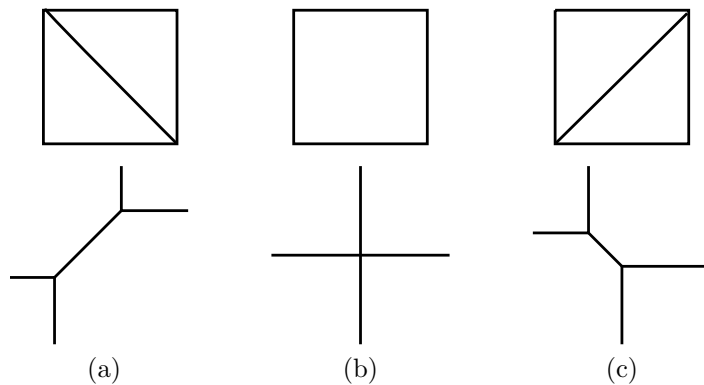

<div align="center">(a)   (b)   (c)</div>

Figure 13: All the possible SNPs, and examples of their corresponding combinatorial types of TH when $A = \{0,1\}^2$.
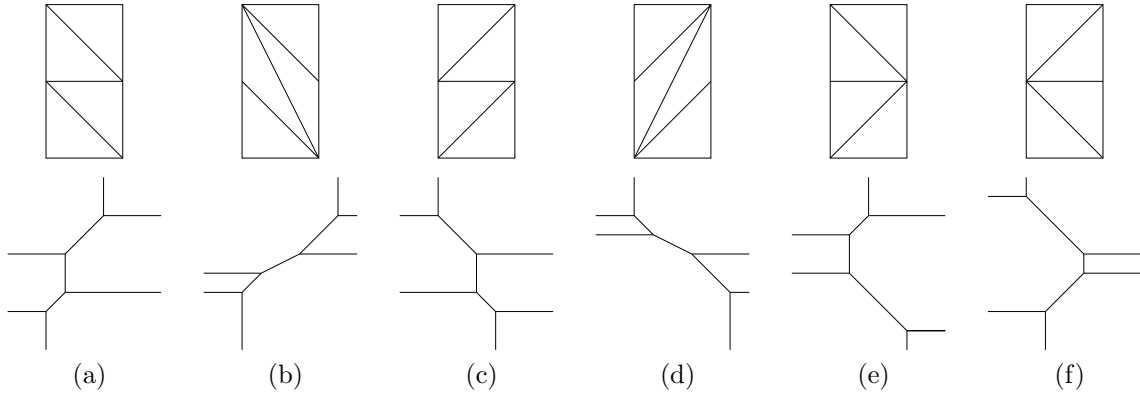
<div align="center">25</div>

Figure 14: All the possible SNPs with maximal subdivision, and examples of their corresponding combinatorial types of TH, when $A = \{0, 1\} \times \{0, 1, 2\}$.

With a bit of practice, starting from either the TH or SNP it is easy to draw the other figure quite fast, at least in two dimensions: if we start with the TH, we know each area around the TH corresponds to a vertex in the SNP, and areas that are separated by a line-segment in the TH correspond to vertices that are connected by a line-segment in the SNP. So we can immediately draw all the vertices and lines. The remaining task is to "straighten out the SNP" without changing it topologically, noting that each line-segment in the SNP is orthogonal to its corresponding line-segment in the TH, and that where a line-segment of weight N is crossed in the TH, there are $(N-1)$ points between the vertices of the corresponding line-segment in the SNP. (The existence of additional points in the SNP that are not on any line segment becomes apparent once the relative positions of all lines are fixed.) Going from the SNP to the TH essentially reverses the process, as we illustrated in Example 3.6, above.

4   Classifying demands: demand "types"

The previous sections suggest classifying demands according to the normal vectors that determine the shapes of agents' THs. We now show that defining demand 'types' in this way does indeed provide a simple characterisation of the standard concepts of substitutes and complements, as well as (in Section 5) more recently developed concepts such as strong substitutes, and gross substitutes and complements, and that demand 'types' also allow us to make other useful distinctions.

We provide a theorem showing how easily a demand type translates to these concepts and, moreover, show how generalisations of the idea of the "single improvement property" (Gul and Stacchetti, 1999), which we will call the "$\mathcal{D}$- and the "$\mathbb{Z}\mathcal{D}$-Improvement Properties", help analyse these distinctions.

Our approach additionally gives a natural answer to the question of when demand "types" are similar: they share many properties when they are unimodular basis changes of each other. Furthermore, we will show in Section 6 that this framework also allows us to develop new results about aggregate demand, for example, about the existence of competitive equilibrium.[38]

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[38]In other work, we use this framework to derive implications about the scope of possible demand

Finally, these results also provide a quick way to check in practical applications (such as in further developments of the Product-Mix Auction) whether demands are, e.g., strong substitutes, or are such that equilibrium exists, since there are easy software solutions to calculate the normal vectors of the TH for any valuation function, u, and hence to immediately reveal the demand's 'type'.

Although we define an agent's demand type by the vectors normal to the facets of its TH (in price space), it would of course be equivalent to define the demand type of an agent by referring to the edges of its SNP (in quantity space). Danilov, Koshevoy and their co-authors focus on quantity space in the course of their impressive body of work that, we will see in Sections 6.3-6.4, has close connections to ours. However, they do not use these vectors to create a taxonomy of demand – we, by contrast, develop a general framework to understand these vectors in economic terms. In particular, as they work almost exclusively in quantity space, they do not see these vectors as giving the changes in demand as we move between generic points in price space (see Theorems 4.4 and 4.5, and Corollary 5.5).[39]

We, however, emphasise price space for several reasons. First, working in price space seems more intuitive and natural. An SNP in quantity space shows (only) the collections of bundles among which the agent is indifferent for some price vectors. By contrast, a corresponding TH in price space shows clearly which bundles are demanded in which regions of prices.[40] So the geometric objects in price space are easier to interpret, and working with them facilitates the development of intuition and understanding.

Second, recall from Theorem 2.8 that any geometric object satisfying the simple 'balancing condition' of Definition 2.7 *is* the TH of some valuation u, but (Fact 3.5) *not* every subdivision of every Newton polytope is induced by some valuation. So we can easily recover the full set of valuations satisfying an additional condition (for example, on their facet normals) from the set of THs in price space, and we can also specify all valuations with a particular property by referring to all THs with the corresponding property in price space–but there are no obvious corresponding procedures to do these things in quantity space.

A further advantage of our approach will become apparent in Section 6.1: it is straightforward to aggregate agents' demands in price space, and it is then also immediately obvious that if two agents have demand of the same demand 'type', then aggregate demand will also be of the same demand 'type'. By contrast, it is not straightforward

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functions which are substitutes; for example, various marginal valuations must be equal. See also Fujishige and Yang (2003).

[39]See Danilov et al. (2001) and Danilov et al. (2003, 2008, 2013). There are also some ways in which our discussion is more general than theirs. Their principal interest corresponds to is what we call 'unimodular demand types' (see Definition 5.9); we explore more general classifications. They focus on examples containing all the coordinate vectors; we see economic interest in valuations that do not satisfy this restriction (see, e.g. Example 2.12). And they assume all bundles in question are non-negative (i.e. $A \subset \mathbb{Z}_+^n$), modelling buyers and sellers separately; by allowing $A \subset \mathbb{Z}^n$ we both simplify the treatment and generalise it, since this allows agents who might both buy and sell.

[40]With n goods, a TH is naturally an $(n-1)$-dimensional object in n-dimensional space, whereas a SNP is best understood as the n-dimensional projection of a collection of related $(n+1)$-dimensional objects. Of course, a specific TH depends on specific details of the valuation, whereas a SNP describes a class of valuations. However, examining any one TH gives the flavour of–and is enough for many purposes to develop intuition and understanding about–the entire class of THs that correspond to any single SNP.

to understand aggregation of agents' demands in quantity space. The reason is that, in price space, the TH of aggregate demand may be understood as simply the union of the THs of individual demands. However, an SNP corresponds only to a combinatorial type of THs–and there is not a unique combinatorial type of THs that corresponds to the aggregate demands formed from individual demands of a set of combinatorial types of THs. So there is no unique way of aggregating SNPs.[41]

4.1 Introducing Demand Types

Let $\mathcal{D} = \{\mathbf{v}^1, \ldots, \mathbf{v}^r\}$ be a set of primitive integer vectors in $\mathbb{Z}^n$, such that if $\mathbf{v} \in \mathcal{D}$ then $-\mathbf{v} \in \mathcal{D}$. (We will often abuse notation by writing the set to include just one representative of each such pair).

Definition 4.1. A valuation is of *demand type* $\mathcal{D}$ if all the primitive integer normals to the facets of the associated TH lie in the set $\mathcal{D}$.

A valuation is of *concave demand type* $\mathcal{D}$ it is of demand type $\mathcal{D}$ and is concave.

The geometric meaning of these sets is that they give the possible slopes of the facets of the THs. But they also have an important economic meaning: recall that each *facet normal gives the direction of change in demand as we cross the facet*. We will see that this combination of being both mathematically tractable and economically intuitive makes them powerful. As noted above, it would be equivalent to define the demand type of an agent by referring to the edges of its SNP (in quantity space). However, price space is in general more intuitive to work with.

We will represent $\mathcal{D}$ by any $n \times r$ matrix D whose columns contain one representative of each $\pm$ pair in $\mathcal{D}$. Of course, D is not unique, since it can include either representative of each $\pm$ pair, and its columns can be in any order, whereas the set $\mathcal{D}$ *is* unique.[42] For example, any of a number of matrices including, for example,

$$\begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & -1 \end{pmatrix}, \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & -1 \end{pmatrix}, \text{ and } \begin{pmatrix} 0 & -1 & -1 \\ -1 & 0 & 1 \end{pmatrix},$$

represent the demand type $\mathcal{D} = \{\pm(1,0), \pm(0,1), \pm(1,-1)\}$ of the THs in Figures 1,

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[41]To aggregate demands in quantity space, we either have to first translate back into price space to perform the aggregation there and then translate back again into quantity space, or–essentially equivalently–we have to find the convolution of the $\widehat{A}$s, which involves considering all possible ways of partitioning any given bundle among the agents. (This is analogous to the standard point that with divisible, uniquely defined demand that it is straightforward to find aggregate demand $\mathbf{q} = \mathbf{F}(\mathbf{p})$ from individual demand functions $\mathbf{q}_i = \mathbf{f}_i(\mathbf{p})$ of agents $i = 1, \ldots, n$ (since $\mathbf{F}(\mathbf{p}) = \sum_i \mathbf{f}_i(\mathbf{p})$), but it is not so easy to find the inverse aggregate demand $\mathbf{p} = \mathbf{G}(\mathbf{q})$ from the individual inverse-demand functions $\mathbf{p} = \mathbf{g}_i(\mathbf{q}_i), i = 1, \ldots, n$.)

[42]Note our definition does not consider the weights on facets. We could take these into account, by relaxing the condition that all vectors in $\mathcal{D}$ be primitive. Then, for every facet F (with weight ω_F and primitive integer normal $\mathbf{v}_F$), we could require either that $\omega_F \mathbf{v}_F \in \mathcal{D}$, or that $\omega_F \mathbf{v}_F = k\mathbf{v}$, for some $k \in \mathbb{Z}$ and some $\mathbf{v} \in \mathcal{D}$. The former approach would allow us to specify the precise weights that facets may possess; this may seem unnatural, since a higher weight facet can be considered as the limit of two lower weight facets as they come arbitrarily close together, and thus very similar agent demands would be classified differently. The latter approach would allow us to insist on certain weak- or non-concavities of demand, and is a more straightforward generalisation of our definition.

4a, 13a, 13b, and 14a.[43] Note that a TH has any demand type which contains its facet normals; we do not restrict to the minimal such set. (So, for example, any of the THs listed earlier in this paragraph are *also* of type $\mathcal{D} \cup \{\pm(N_1, N_2)\}$, for any primitive integer (N_1, N_2).)

The set of vectors in a demand type determines the complete set of ways in which demand can change as we move between adjacent UDRs, and thus the possible changes in demands that can generically result from a small change in prices (since the UDRs are dense in price space). So we can immediately identify properties such as substitutes or complements.

For example, in Figure 4a–the simplest case of substitutes–an increase in any good's price that moves between UDR's can result in the agent swapping that good for another good, but can never result in the agent decreasing its demand for another good. That is, if the demand for one good changes when its own price does, then the change in demand for another good cannot be in the same direction–and this is precisely reflected in the fact that the vectors that are normal to the facets may have two non-zero entries of opposite signs, but never have two non-zero entries of the same sign. Likewise, in Figure 4b–the simplest case of complements–if either good's price increases to move across the downward-sloping diagonal facet, then the agent reduces its quantity of both goods, precisely because both components of the vector normal to this facet are of the same sign. Moreover, in this case there is no facet whose normal vector has two non-zero entries of different signs.

So we can distinguish between substitutes and complements valuations by examining the coordinate entries of the vectors in demand types. But to make the distinctions precise and, in particular, to deal with some subtleties involving changes in prices at which demand is non-unique, it is helpful to first develop the additional analytical tool of "$\mathbb{Z}\mathcal{D}$ steps".

4.2 $\mathcal{D}$- and $\mathbb{Z}\mathcal{D}$-Steps, and the $\mathcal{D}$- and $\mathbb{Z}\mathcal{D}$-Improvement Properties

We now show that the change in demand along the (straight) line joining any pair of prices at which demands are unique can be broken down into changes that are 'improving $\mathbb{Z}\mathcal{D}$-*steps*', and the change in demand from the first price to the last price satisfies the 'ordinary $\mathbb{Z}\mathcal{D}$-improvement property'.

These concepts are introduced to show that 'demand types' generalise the way in which we normally think about, for example, a substitutes valuation. If prices change from $\mathbf{p}$ to $\mathbf{p}' \geq \mathbf{p}$, then we would think of an agent as viewing these goods as substitutes if their demand weakly increases for all goods whose price has not changed. When goods are indivisible, their demand will change from that at $\mathbf{p}$ to that at $\mathbf{p}'$ in discrete steps. By understanding the nature of these steps, we may understand all ways in which demand may change for a valuation with the property we are studying.

There are two ways to construct such steps. We may either consider a gradual change in price on the straight line from $\mathbf{p}$ to $\mathbf{p}'$, and look at the step changes in demand that will be triggered *en route.* Or we may think of the price change as having taken place

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[43]We will see later (Section 5.3) that this demand type is "strong substitutes" in the two-good case, which we will label $\mathcal{D}_{ss}^2$.

all in one go, but that the agent is incrementally swapping bundles for ones that are preferred at the new and final prices. In either case, if the agent sees goods as (for example) 'substitutes', we expect this property to be evident at every one of the steps (just as, with divisible demand, the corresponding property is true at all prices).

By treating such steps as our primitives, we can build up a more general understanding of the economic nature of valuations for goods, be they substitutes, complements, or some combination.

By '$\mathbb{Z}\mathcal{D}$-steps', then, we mean a sequence of bundles demanded on the straight line between $\mathbf{p}$ and $\mathbf{p}'$ such that the differences between consecutive bundles in this sequence are vectors in $\mathcal{D}$ or integer multiples thereof (we denote these vectors as $\mathbb{Z}\mathcal{D} := \{w\mathbf{v} \mid w \in \mathbb{Z}, \mathbf{v} \in \mathcal{D}\}$); by 'improving', we mean that the bundle at the end of each step is preferred to the bundle at the beginning of each step, at the final price vector.

It follows immediately that *any* valuation of *any* demand type $\mathcal{D}$ must satisfy a property we will call the '*ordinary $\mathbb{Z}\mathcal{D}$-improvement property*': given any starting bundle, $\mathbf{x}$, which is the unique demand at some price, $\mathbf{p}$, and any price $\mathbf{p}'$ at which $\mathbf{x}$ is not demanded, there exists a bundle $\mathbf{x}''$ which is strictly preferred to $\mathbf{x}$ at $\mathbf{p}'$, and such that $\mathbf{x}'' - \mathbf{x} \in \mathbb{Z}\mathcal{D}$.[44]

This property is closely connected to Gul and Stacchetti's (1999) result that a specific set of valuations satisfies their "single improvement property".[45] This holds if, given any starting bundle, $\mathbf{x}$, and any price $\mathbf{p}'$ at which $\mathbf{x}$ is not demanded, there exists a bundle $\mathbf{x}''$ which is strictly preferred to $\mathbf{x}$ at $\mathbf{p}'$, and such that $\mathbf{x}'' - \mathbf{x}$ is a vector whose entries include include at most one $+1$, at most one -1, and all others zero. Since the ordinary $\mathbb{Z}\mathcal{D}$-improvement property applies only to "starting bundles", $\mathbf{x}$, corresponding to UDR prices, it is not a strict generalisation of the "single improvement property" which applies to *all* starting bundles for those valuations for which it holds. However, we will introduce a refinement of our property which strictly generalises the "single improvement property", and so allows strict generalisations of Gul and Stacchetti's (1999) results, in Section 5.

The importance of our definitions and results is that we can use them to show how our "demand types" correspond precisely to interesting properties of demand such as whether agents view goods as substitutes or complements; see Subsections 4.3.1 and 4.3.2.

We proceed by first giving a formal definition of improving $\mathbb{Z}\mathcal{D}$-steps that is easily shown to be equivalent to the informal definition given above (see the discussion of Theorem 4.4), and will be easier to work with. Specifically we will require that an *improving $\mathbb{Z}\mathcal{D}$-step* is a $\mathbb{Z}\mathcal{D}$-step which "satisfies the strict law of demand" with respect to the *overall* price change:[46]

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[44]It obviously suffices to let $\mathbf{x}''$ be the bundle at the end of the first of any set of improving $\mathbb{Z}\mathcal{D}$-steps into which the demand change is broken down. Note, however, that the way we define our 'ordinary $\mathbb{Z}\mathcal{D}$-improvement property' does *not* require that $\mathbf{x}''$ be demanded at some price on the straight line between $\mathbf{p}$ and $\mathbf{p}'$. We do this for consistency with Gul and Stacchetti's (1999) related definition (see below). But the stronger concept of a $\mathbb{Z}\mathcal{D}$-step will be easier to use to characterise properties that valuations may have.

[45]Gul and Stacchetti (1999) restrict their attention to $A = \{0, 1\}^n$. We will see later (Section 5.3) that this set of valuations corresponds to our concave demand type $\mathcal{D}_{ss}^n$.

[46]That is, for an *overall* price change from $\mathbf{p}$ to $\mathbf{p}'$, and a demand *step* from $\widetilde{\mathbf{x}}$ to $\widetilde{\widetilde{\mathbf{x}}}$, we require $(\mathbf{p}' - \mathbf{p}).(\widetilde{\widetilde{\mathbf{x}}} - \widetilde{\mathbf{x}}) < 0$, with the exception that the change in demand is, of course, zero if $\mathbf{p}$ and $\mathbf{p}'$ are

Definition 4.2. For prices $\mathbf{p}$ and $\mathbf{p}'$ such that $D_u(\mathbf{p}) \neq D_u(\mathbf{p}')$, we say we can *break down the demand change from* $\mathbf{p}$ *to* $\mathbf{p}'$ *into* $\mathbb{Z}\mathcal{D}$-*steps* if for all $\mathbf{x} \in D_u(\mathbf{p})$ there exists $\mathbf{x}' \in D_u(\mathbf{p}')$, and a series of bundles $\mathbf{x} = \mathbf{x}^0, \ldots, \mathbf{x}^l = \mathbf{x}'$, demanded respectively at prices $(1-\lambda_j)\mathbf{p} + \lambda_j \mathbf{p}'$, $j = 0, \ldots, l$, for some $0 = \lambda_0 \leq \cdots \leq \lambda_l = 1$, such that $\mathbf{x}^j - \mathbf{x}^{j-1} \in \mathbb{Z}\mathcal{D}$.

We call these $\mathbb{Z}\mathcal{D}$-steps *improving* if each demand change additionally "satisfies the strict law of demand" with respect to the *whole* price change, that is, $(\mathbf{p}' - \mathbf{p}).(\mathbf{x}^j - \mathbf{x}^{j-1}) < 0$.

We say we can *break down the demand change into improving* $\mathcal{D}$-*steps* if there always exists $\mathbf{x}^0, \ldots, \mathbf{x}^l = \mathbf{x}' \in D_u(\mathbf{p}')$ satisfying these conditions and $\mathbf{x}^j - \mathbf{x}^{j-1} \in \mathcal{D}$ for all j.

Definition 4.3. We say a valuation u satisfies the *ordinary* $\mathbb{Z}\mathcal{D}$-*improvement property* if, for any bundle $\mathbf{x}$ which is the unique demand at some price, $\mathbf{p}$, i.e., satisfying $D_u(\mathbf{p}) = \{\mathbf{x}\}$, and any price $\mathbf{p}'$ such that $\mathbf{x} \notin D_u(\mathbf{p}')$, there exists $\mathbf{x}''$ which is strictly preferred to $\mathbf{x}$ at price $\mathbf{p}'$, and such that $\mathbf{x}'' - \mathbf{x} \in \mathbb{Z}\mathcal{D}$.

We say the valuation u satisfies the *ordinary* $\mathcal{D}$-*improvement property* if there always exists $\mathbf{x}''$ which is strictly preferred to $\mathbf{x}$ at $\mathbf{p}'$ and $\mathbf{x}'' - \mathbf{x} \in \mathcal{D}$.

Theorem 4.4. *If* $\mathcal{D}$ *is any demand type, the following are equivalent for a valuation* u:

1. *u is of demand type $\mathcal{D}$;*

2. *for any $\mathbf{p}$ such that $\#D_u(\mathbf{p}) = 1$ and any $\mathbf{p}'$ we can break down the demand change from $\mathbf{p}$ to $\mathbf{p}'$ in improving $\mathbb{Z}\mathcal{D}$-steps;*

3. *for any $\mathbf{p}$ such that $\#D_u(\mathbf{p}) = 1$, and any $i \in \{1, \ldots, n\}$ and any $\epsilon > 0$, we can break down the demand change from $\mathbf{p}$ to $\mathbf{p} + \epsilon \mathbf{e}^i$ in $\mathbb{Z}\mathcal{D}$-steps;*

4. *u satisfies the ordinary $\mathbb{Z}\mathcal{D}$-improvement property.*

Proof. See Appendix A.1. $\qquad\square$

Condition 2 implies 4 straightforwardly, because the fact that each separate step satisfies the "strict law of demand" with respect to the overall price change, means that each successive bundle in the sequence must be strictly preferred to its predecessor at the final price. So, in particular, the second bundle in the sequence is a bundle that is strictly preferred to the starting bundle at the final price. Condition 4 implies 1 because the only bundles preferred to a bundle demanded in a UDR, at any price that is just on the other side of any facet bounding the UDR, are demanded on the facet itself, so the ordinary $\mathbb{Z}\mathcal{D}$-improvement property implies that the primitive integer facet normal must lie in $\mathcal{D}$. Condition 2 clearly implies 3, and 3 implies 1 because any facet whose normal was not in $\mathcal{D}$ would not satisfy property 3 for some good i. Finally, the relationship between 1 and 2 is straightforward for any pair of prices at which the demands are unique, and for which the straight line joining them crosses only facet interiors, although it needs more careful argument when the price path crosses lower dimensional cells of the TH.

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within the same UDR. (The strict law of demand generalises the observation that demand must go down for goods whose prices go up–see e.g. Mas-Colell et al., 1995, Proposition 2.F.1.)

The inclusion of 3 shows that studying demand changes arising from a change in a single price suffices to understand the shapes of the trade-offs an agent might make and, moreover, that we need not assume the 'improving' property, since this will hold automatically.

It is, however, very useful to know that we can always choose 'improving' $\mathbb{Z}\mathcal{D}$-steps, since this greatly restricts the range of possible steps for any price change. We will see this when we prove the relationships between certain demand types and the concepts of substitutes and complements (see Propositions 4.8 and 4.11).

The reason we need $\mathbb{Z}\mathcal{D}$-steps, rather than only $\mathcal{D}$-steps, is because of possible failures of concavity–consider, for example, the function of one variable $v(0) = v(1) = 0$, $v(2) = 2$ for which demand jumps from zero to 2 as price rises above 1. If we additionally assume that u is concave throughout, we can strengthen Theorem 4.4 to deal with improving $\mathcal{D}$-steps and the ordinary $\mathcal{D}$-improvement property:

Theorem 4.5. *If $\mathcal{D}$ is any concave demand type, the following are equivalent for a concave valuation u:*

1. *u is of (concave) demand type $\mathcal{D}$;*

2. *for any $\mathbf{p}$ such that $\#D_u(\mathbf{p}) = 1$ and any $\mathbf{p}'$ we can break down the demand change from $\mathbf{p}$ to $\mathbf{p}'$ in improving $\mathcal{D}$-steps;*

3. *for any $\mathbf{p}$ such that $\#D_u(\mathbf{p}) = 1$, and any $i \in \{1, \dots, n\}$ and any $\epsilon > 0$, we can break down the demand change from $\mathbf{p}$ to $\mathbf{p} + \epsilon\mathbf{e}^i$ in improving $\mathcal{D}$-steps;*

4. *u satisfies the ordinary $\mathcal{D}$-improvement property.*

Proof. See Appendix A.1. □

On the other hand, we might ask whether concavity need be explicitly assumed in 2, 3 and 4. However, Example A.1 in the appendix shows the fact that we can always break down the demand change between any UDR prices in improving $\mathcal{D}$-steps does not imply u is concave, so we must assume concavity for 2 and 3. Similarly, Example A.2 shows that the ordinary $\mathcal{D}$-improvement property on its own does not imply concavity. Thus Theorem 4.5 is indeed stated in its most general form.

We now illustrate the application of improving $\mathbb{Z}\mathcal{D}$-steps, and Theorem 4.4, in the cases of 'ordinary' substitutes and complements:

4.3 Examples

4.3.1 Ordinary Substitutes

We use the standard definitions of "(ordinary) substitutes" as in Ausubel and Milgrom (2002).[47] An appealing aspect of this definition is that, as they show (their Theorem 10), it is equivalent to the submodularity of the dual profit function.

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[47]That is, we call "ordinary substitutes", precisely what Ausubel and Milgrom (2002) simply call "substitutes". We hope this increases clarity (since others loosely refer to substitutes in other ways). Note, in particular, that Ausubel and Milgrom's (2002) definition (our Definition 4.6) is *not* identical to that of Kelso and Crawford (1982) when there are multiple units of three or more goods. (See Danilov et al. 2003 Example 6 and Theorem 1); the definitions *are* equivalent in the simpler cases $n = 2$ (see

Definition 4.6 (Ausubel and Milgrom, 2002)**.** Let $A \subsetneq \mathbb{Z}^n$ be finite, and $u : A \to \mathbb{R}$ be a valuation. Goods are *ordinary substitutes* if for any prices $\mathbf{p}' \geq \mathbf{p}$ such that $\#D_u(\mathbf{p}) = \#D_u(\mathbf{p}') = 1$, if $\{\mathbf{x}\} = D_u(\mathbf{p})$ and $\{\mathbf{x}'\} = D_u(\mathbf{p}')$ then $x'_k \geq x_k$ for all k such that $p_k = p'_k$.

We define correspondingly the demand type, $\mathcal{D}^n_{os}$.

Definition 4.7. $\mathcal{D}^n_{os}$ consists of those primitive integer vectors in $\mathbb{Z}^n$ with at most one positive and at most one negative coordinate entry, and all others zero.[48]

Theorem 4.4 enables us to easily relate $\mathcal{D}^n_{os}$ to ordinary substitutes; we retain the proof in the body text as it provides a powerful illustration of how useful it is to 'break down demand changes in improving $\mathbb{Z}\mathcal{D}$-steps'.[49]

Proposition 4.8. *A valuation is of demand type $\mathcal{D}^n_{os}$ iff it is an ordinary substitutes valuation.*

Proof. We apply Theorem 4.4, as follows. If valuation is of demand type $\mathcal{D}^n_{os}$ then we can break down the demand change from any UDR price $\mathbf{p}$ to any UDR price $\mathbf{p}' \geq \mathbf{p}$ in improving $\mathbb{Z}\mathcal{D}$-steps. Since each step is in $\mathcal{D}^n_{os}$, demand strictly decreases for at most one good at each step, and by the strict law of demand, that good must be one whose price has increased. Thus at each step, demand weakly increases for all goods whose prices have remained constant. Hence this holds overall.

Conversely, if the valuation is not of demand type $\mathcal{D}^n_{os}$ then there exists a facet F with normal $\mathbf{v}$ where $v_i, v_j < 0$ for some $i \neq j$. Then $\mathbf{e}^i.\mathbf{v} \neq 0$ so we may choose UDR prices $\mathbf{p}, \mathbf{p} + \epsilon\mathbf{e}^i$ on either side of this facet. We know the demand change is a positive integer multiple of $\mathbf{v}$, and so demand for good j decreases: goods i and j are not substitutes. $\qquad\square$

So we can straightforwardly identify whether goods are ordinary substitutes from their 'demand type'.

It is immediate, for example, that the examples of Figures 1, 4a, 5, 13a, 13b, 14a and 14b, are all of type $\mathcal{D}^2_{os}$, as is Example 2.10 (Figure 3), while our 3-dimensional example, Figure 6, has demand type $\mathcal{D}^3_{os}$, and we now show Example 2.14 and Example 2.15 have demand type $\mathcal{D}^n_{os}$.

Baldwin, Klemperer and Milgrom, in preparation) and $A = \{0, 1\}^n$ (Danilov et al. 2003, Corollary 5; see also Hatfield et al. 2011 Theorem A.1). Milgrom and Strulovici (2009) call Kelso and Crawford's original definition "weak substitutes", but this is in fact a *stronger* definition of substitutes than Ausubel and Milgrom's. The latter definition (that we follow) seems most natural in the general case, and is also equivalent to several properties that seem to naturally characterise "standard" substitutes, and to the indirect utility function ($\max_{\mathbf{x} \in A}\{u(\mathbf{x}) - \mathbf{p}.\mathbf{x}\}$) being submodular. We discuss these issues in detail in Baldwin, Klemperer and Milgrom (in preparation).

[48]Danilov et al. (2003) say 'each cell of a valuation's parquet is a polymatroid' where we say that a valuation has demand type $\mathcal{D}^n_{os}$.

[49]This result also follows if we combine Ausubel and Milgrom (2002, Theorem 10) and Danilov et al. (2003, Theorem 1); we provide this alternative proof to illustrate the use of Theorem 4.4 in understanding demand types.

Example 2.14 revisited. Because the valuations of the 'workers' in Kelso and Crawford's (1982) matching model have domain $\{\mathbf{0}, -\mathbf{e}^{(i,j)} \mid j = 1, \ldots, m_2\}$, their only possible SNP edges are in $\mathcal{D}_{os}^n$, so they are of demand type $\mathcal{D}_{os}^n$. Since the 'firms' in this model have valuations with domain $\{0,1\}^{m_1}$ which are assumed to satisfy the conditions for ordinary substitutes, they are also of demand type $\mathcal{D}_{os}^n$.

Example 2.15 revisited. Hatfield et al. (2013) describe the goods to be sold in their model as *complements* of goods to be bought, because they measure both buying and selling as non-negative quantities. But if we instead think of selling as just "negative buying" then the "complementarities" in their model disappear, and it is then clear that their condition of 'full substitutability' is precisely the ordinary substitutes condition of, e.g., Ausubel and Milgrom (2002). That is, an agent whose valuation domain is as described in Example 2.15 has 'fully substitutable' preferences iff the valuation is of type $\mathcal{D}_{os}^n$.[50]

Example 2.12 illustrates a special case of this model: the agent regards the actions of buying good 2 and selling good 1 as complements, but regards the *buying* of both goods (possibly in negative quantities) as substitutes. The point is easily seen geometrically– Figure 5 clearly represents substitutes preferences, and not the complements preferences of Figure 4b, which we formally introduce in Section 4.3.2.

4.3.2 Ordinary Complements

"Complements" can be defined analogously to the Definition 4.6 of "ordinary substitutes":

Definition 4.9. Let $A \subsetneq \mathbb{Z}^n$ be finite, and let $u : A \to \mathbb{R}$ be a valuation. Goods are *ordinary complements* if, for any prices $\mathbf{p}' \geq \mathbf{p}$ such that $\#D_u(\mathbf{p}) = \#D_u(\mathbf{p}') = 1$, if $\{\mathbf{x}\} = D_u(\mathbf{p})$ and $\{\mathbf{x}'\} = D_u(\mathbf{p}')$ then $x_k' \leq x_k$ for all k such that $p_k = p_k'$.

Similarly to Definition 4.7 we define a corresponding demand type:

Definition 4.10. $\mathcal{D}_{oc}^n$ consists of those primitive integer vectors in $\mathbb{Z}^n$ whose non-zero coordinate entries are all of the same sign.

As in Proposition 4.8, it is an elementary consequence of Theorem 4.4 that

Proposition 4.11. *A valuation is of demand type $\mathcal{D}_{oc}^n$ iff it is an ordinary complements valuation.*

Proof. As with Proposition 4.8, this follows immediately from Theorem 4.4: we break down the change in demand from some UDR price $\mathbf{p}$ to $\mathbf{p}' \geq \mathbf{p}$ into improving $\mathbb{Z}\mathcal{D}$-steps. As before, at each step demand must strictly decrease for at least one good whose price has increased; this time, the nature of $\mathcal{D}_{oc}^n$ implies that demand weakly decreases for

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[50]The precise 'choice language' definition of Hatfield et al. (2013) is superficially different from our Definition 4.6, but an earlier version of their paper presents an alternative 'demand-language' definition (Hatfield et al., 2011, Definition 4) which corresponds precisely to the Ausubel and Milgrom (2002) definition, and moreover they confirm (Hatfield et al., 2011, Theorem A.1) that this definition is equivalent to the 'choice-language' definition of Hatfield et al. (2013). They define "full substitutability" using Sun and Yang's (2006) 'gross substitutes and complements' ideas; see Section 5.5.
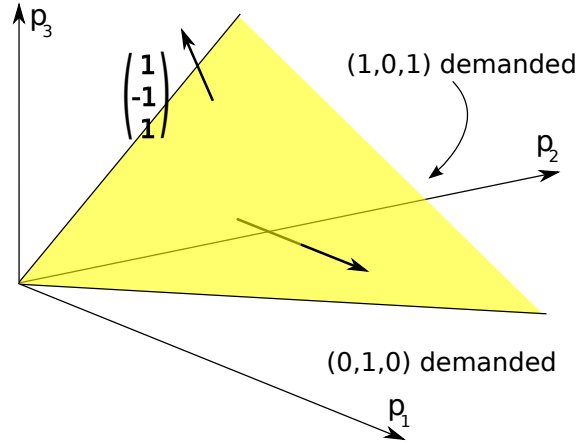
Figure 15: A facet with normal $(1, -1, 1)$: increasing either p_1 (as shown with an arrow) or p_3 demonstrates complementarities between goods 1 and 3.

all goods. Conversely, if there exists a facet with normal $\mathbf{v}$ such that $v_i > 0, v_j < 0$ then we may pick prices $\mathbf{p}, \mathbf{p} + \epsilon \mathbf{e}^j$ on either side of this facet and demonstrate failure of complements. $\qquad \square$

The examples of Figures 4b, 13b, 13c, 14c and 14d are all of type $\mathcal{D}_{oc}^2$.

Note that although complements are often thought of as directly analogous to (ordinary) substitutes–as they are in two dimensions–this is not true if there are more than two goods. The case of complements permits facet normals with *any* number of non-zero entries, whereas substitutes permits at most two non-zero entries.

The reason is that with substitutes, if any one good could trade-off against two others at the same price, it would necessarily follow that the two other goods were complementary. Even when all goods are mutual substitutes, there can never be trade-offs between more than two of them across a single facet: if more than two facet normal coordinate entries are non-zero, then at least two must have the same sign, so there are complementarities between the corresponding goods.

Consider, for example, Figure 15, in which there is a facet with normal $(1,-1,1)$, defined by $\{\mathbf{p} \in \mathbb{R}^3 \mid p_1 + p_3 = p_2; p_1, p_2, p_3 \geq 0\}$: an increase in the price of *either* good 1 or good 3 that moves from the UDR with $p_1 + p_3 < p_2$ to the UDR with $p_1 + p_3 > p_2$ reduces demand for *both* goods. So, despite the symmetry between Definitions 4.6 and 4.9, complements allows far more degrees of freedom than does substitutes.[51] One benefit of our way of classifying demand "types" is that it makes this lack of symmetry between substitutes and complements very clear.

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[51]To illustrate why the conditions for indivisible goods to be substitutes are so restrictive, consider a consumer who regularly makes three kinds of trips: journey A can be made only by bus or train; journey B can be made only by car or train; journey C can be made only by car or bus. Thought of as divisible goods, the three modes of transport are clearly always mutual substitutes. But if bus tickets, train tickets, and cars are all *in*divisible, there are typically price vectors at which two of the goods are locally complements. Start at any prices at which the consumer just prefers to use only public transport, i.e., has no car. Then if the price of *either* of the forms of public transport is slightly raised, the consumer buys a car and in general reduces her use of *both* forms of public transport. Qualitatively the situation is locally exactly that pictured in Figure 15, in which the car takes the role of good 2, and the two forms of public transport take the roles of goods 1 and 3.

Example 2.16 revisited. Recall that we can embed a model of 'coalition formation with transferable utilities' in our framework by considering each feasible coalition as a separate agent. The agent corresponding to any feasible coalition $\mathbf{x}$ has valuation with domain $\{\mathbf{0}, \mathbf{x}\}$, so the only facet of the corresponding TH has normal $\mathbf{x}$: the agent views the people who would form this coalition as perfect complements, and considers only the trade-off between all of them and none of them.

Consistent with our terminology, we say a coalition-formation problem with transferable utility is of 'type $\mathcal{D}$', for any $\mathcal{D}$ containing all the feasible coalitions in the problem. (We would normally consider 'complements' demand types, $\mathcal{D}$.)

4.3.3 Additively Separable Demand

Additively separable demand corresponds to an extremely simple demand type:

Definition 4.12. $\mathcal{D}_a^n$ consists of the coordinate vectors $\{\mathbf{e}^i \mid i = 1, \ldots, n\}$ in $\mathbb{Z}^n$.

In the additively separable case, a change in the price of one good will never affect demand for any other good. So it is not hard to show:

Proposition 4.13. *A valuation is of concave demand type $\mathcal{D}_a^n$ iff it is concave and additively separable.*

Proof. Recall that demand is additively separable iff a change in the price for any one good has no effect on the demand for other goods; if demand is additionally concave, then it is also true that demand is additively separable iff a change in price between UDR prices for any one good has no effect on the demand for other goods. Referring to Theorem 4.4, we see that this holds iff the change in demand at each step must only affect one good – that is, $\mathcal{D} = \mathcal{D}_a^n$. □

Note that being additively separable is a more stringent condition than being both substitutes and complements: we can only guarantee such a valuation is additively separable if it is also concave. A simple example of a valuation of type $\mathcal{D}_a^2$ which is not concave, and not additively separable is: $A = \{0, 1, 2\}^2$, and

$$u(x_1, x_2) = \left\{ \begin{array}{ll} x_1 + x_2 & (x_1, x_2) \neq (1, 1) \\ 0 & (x_1, x_2) = (1, 1). \end{array} \right.$$

4.4 Changes of basis

It is straightforward that two demands share many properties if one can be transformed into the other by a unimodular basis change.[52,53] Such a basis change is equivalent to re-packaging the goods so that any integer bundle can still be obtained by buying and selling an (integer) selection of the new packages; and any integer selection of the new packages was available as an integer combination of the original goods. So such a basis

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[52]A unimodular matrix G is an integer matrix with integer inverse; an action of G on bundles of goods corresponds to an action of G^T on prices.

[53]Specific cases of this observation have been made before (see e.g. Sun and Yang 2006, and a more general treatment for substitutes in Sun and Yang 2008, and Hatfield et al., 2013); we lay out the general behaviour here.

change leaves many important properties of demand–including, we will see in Section 6, the existence of competitive equilibrium–unaffected.

Likewise, such a basis change simply distorts the TH by a linear transformation which leaves its important structure unaffected:

Proposition 4.14 (cf. Gorman, 1976, p. 219-20). *For $A \subsetneq \mathbb{Z}^n$ and $u : A \to \mathbb{R}$ and a unimodular $n \times n$ matrix G, define the ("pullback") basis change of u by G to be $G^*u : G^{-1}A \to \mathbb{R}$ via $G^*u(\mathbf{y}) := u(G\mathbf{y})$. Then*

1. *A bundle is demanded under the original demand at a certain price iff an associated bundle is demanded under the transformed demand at an associated price; specifically: $\mathbf{x} \in D_u(\mathbf{p}) \iff G^{-1}\mathbf{x} \in D_{G^*u}(G^T\mathbf{p})$.*

2. *The TH of the transformed demand is given by a linear transformation of the original demand: $\mathcal{T}_{G^*u} = \{G^T\mathbf{p} \mid \mathbf{p} \in \mathcal{T}_u\}$;*

3. *The inverse transformation to G applies to demand types: $u(\cdot)$ is of (concave) demand type $\mathcal{D}$ iff $G^*u(\cdot)$ is of (concave) demand type $G^{-1}\mathcal{D} = \{G^{-1}\mathbf{v} \mid \mathbf{v} \in \mathcal{D}\}$.*

Proof. See Appendix A.1. □

As a simple example, if $n = 2$ then ordinary substitutes $\mathcal{D}_{os}^n$ are a unimodular basis change of ordinary complements $\mathcal{D}_{oc}^n$, via the matrix $\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$.[54] As discussed in Section 4.3.2, this does not hold for $n \geq 3$.

5 Complete Demand Types and Unimodular Demand Types

The previous section showed that the change in demand between any two UDR prices could be broken down into improving $\mathbb{Z}\mathcal{D}$-steps and, for concave valuations, into improving $\mathcal{D}$-steps. Since UDR prices are dense in the set of all prices, this tells us a great deal about the structure of demand.

However, one may wish for results relating to every possible price and starting bundle. For example, the standard definition of 'ordinary substitutes' for indivisible goods (see, e.g., Ausubel and Milgrom, 2002) that we use only considers demand changes between UDR prices, but others have used a definition of substitutes that compares demands at any pair of prices (see, for example, Kelso and Crawford 1982).[55] As is shown by Danilov et al. (2003, Example 6), for $n \geq 3$ there exist valuations which satisfy the 'ordinary

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[54]So, for example, we will see that with two goods (in indivisible units), competitive equilibrium fails "as often" for sets of agents with ordinary substitutes demands, as for sets of agents with ordinary complements demands, even though the economic properties of substitutes and complements are, of course, very different.

[55]These two definitions are equivalent when $A = \{0, 1\}^n$ (see Corollary 5.17) and almost all preceding work has been restricted to this case, so the distinction between these definitions is sometimes blurred. We discuss the relationships between alternative definitions of substitutes at length in Baldwin, Klemperer and Milgrom (in preparation).

substitutes' property but do not satisfy the stronger requirement. Similar remarks apply to 'ordinary complements'.

In this section we clarify this distinction, and introduce a property which a valuation might have: $\mathcal{D}$-completeness, which means we that *can* break down the change in demand between any two prices into improving $\mathcal{D}$-steps. So if a valuation is $\mathcal{D}_{os}^n$- or $\mathcal{D}_{oc}^n$-complete then it does indeed satisfy the corresponding stronger substitute or complement conditions.

We correspondingly identify a new class of 'complete' demand types: those types $\mathcal{D}$ for which every valuation of type $\mathcal{D}$ is $\mathcal{D}$-complete.

We begin with an illustration of a demand type $\mathcal{D}$ and a concave valuation of type $\mathcal{D}$ which is *not* $\mathcal{D}$-complete: we cannot break down the change in demand between any two prices into improving $\mathcal{D}$-steps, or even $\mathbb{Z}\mathcal{D}$-steps, even for concave valuations:[56]

Example 5.1. Consider the following (concave) valuation given, with its SNP, in Figure 16. This is of demand type $\mathcal{D} = \{\pm(1,0), \pm(0,1), \pm(2,1)\}$. Note that $(1,1) \in D_u(2,3)$

$x_1 = 2$	$x_1 = 1$	$x_1 = 0$	u
2	1	0	$x_2 = 0$
7	5	3	$x_2 = 1$
10	7	4	$x_2 = 2$

(a) The concave valuation u.

(b) The SNP of u.

(c) The TH of u.
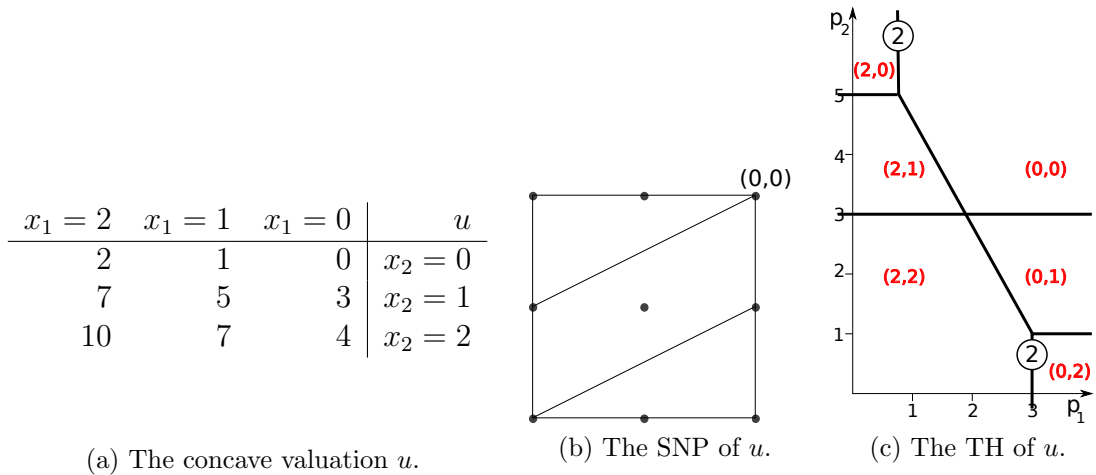
Figure 16: The valuation, SNP, and TH of Example 5.1.

but that $D_u(2, 3 - \epsilon) = \{(2,2)\}$ for any $\epsilon > 0$. So any series of improving $\mathbb{Z}\mathcal{D}$-steps from the bundle $(1,1)$ demanded at $\mathbf{p} = (2,3)$, to the unique bundle, $(2,2)$, demanded at $\mathbf{p}' = (2, 2.9)$, would have to increase the demand for the first good, as well as the second – but a step in direction $(1,0)$ does not satisfy the strict law of demand for the price change we consider, and a step in direction $(2,1)$ increases demand for the first good by too much. Thus we cannot break down the change in demand between these prices in improving $\mathbb{Z}\mathcal{D}$-steps.

We now turn to developing exactly what we mean by 'completeness':

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[56]When $n = 2$ the ordinary substitutes demand type $\mathcal{D}_{os}^n$ is in fact 'complete', in our terminology (see Baldwin, Klemperer and Milgrom, in preparation). The 3-dimensional example of Danilov et al. (2006) is more complicated than is needed here, and so we use a very restrictive demand type.

5.1 Completeness of a Demand Type

Recall that Definition 4.3 was not exactly a generalisation of Gul and Stacchetti's (1999) "single-improvement property": a true generalisation may be given as follows:[57]

Definition 5.2. We say a valuation u satisfies the *complete $\mathcal{D}$-improvement property* if, for any bundle $\mathbf{x}$ and for any price $\mathbf{p}'$ such that $\mathbf{x} \notin D_u(\mathbf{p}')$, there exists $\mathbf{x}''$ which is strictly preferred to $\mathbf{x}$ at price $\mathbf{p}'$ and such that $\mathbf{x}'' - \mathbf{x} \in \mathcal{D}$.

Note how this differs from Definition 4.3: for the 'complete' property we do not insist that the bundle $\mathbf{x}$ be demanded at any price at all, whereas the 'ordinary' property requires that $\mathbf{x}$ be demanded *uniquely* at some price.

We show that, given concavity, the complete $\mathcal{D}$-improvement property is equivalent to our being able to break down any change in demand into improving $\mathcal{D}$-steps:

Proposition 5.3. *The following are equivalent for a concave valuation u of type $\mathcal{D}$:*

1. *we can break down the demand change from any $\mathbf{p}$ to any $\mathbf{p}'$ in improving $\mathcal{D}$-steps;*

2. *u satisfies the complete $\mathcal{D}$-improvement property.*

Proof. See Appendix A.2. ☐

The assumption of concavity is needed to ensure that every bundle $\mathbf{x}$ is demanded at *some* price – and hence that we can use improving $\mathcal{D}$-steps to obtain $\mathcal{D}$-improvements.[58] Since concavity is also needed for the existence of improving $\mathcal{D}$-steps and the ordinary $\mathcal{D}$-improvement property (as distinct from $\mathbb{Z}\mathcal{D}$-steps and the ordinary $\mathbb{Z}\mathcal{D}$-improvement property) we simply work under this assumption for this section.

Definition 5.4.

1. We say a concave valuation u is *$\mathcal{D}$-complete* if the equivalent conditions of Proposition 5.3 hold.

2. We say a concave demand type $\mathcal{D}$ is a *complete* demand type if every concave valuation of type $\mathcal{D}$ is $\mathcal{D}$-complete.

It is straightforward now to provide the following analogue to Theorems 4.4 and 4.5:

Corollary 5.5. *If $\mathcal{D}$ is any complete demand type, then the following are equivalent for a concave valuation u:*

1. *u is of concave demand type $\mathcal{D}$ (and the equivalent conditions of Theorem 4.4 hold);*

2. *we can break down the demand change from any $\mathbf{p}$ to any $\mathbf{p}'$ in improving $\mathcal{D}$-steps;*

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[57]This also generalises Sun and Yang (2009), which generalised Gul and Stacchetti (1999) to the case of "gross substitutes and complements".

[58]In fact, 5.3.2⇒5.3.1 when u is not concave; one may see that the proof makes no use of concavity. However, a non-concave valuation satisfying 5.3.1 need not satisfy 5.3.2; consider Example 5.1 with the modification $u(1,1) = 4$.

3. *for any* $\mathbf{p}$ *and any* $i \in \{1, \ldots, n\}$ *and any* $\epsilon > 0$, *we can break down the demand change from* $\mathbf{p}$ *to* $\mathbf{p} + \epsilon \mathbf{e}^i$ *in improving* $\mathcal{D}$*-steps;*

4. u *satisfies the complete* $\mathcal{D}$*-improvement property.*

Proof. Given that $\mathcal{D}$ is complete, 1⇔2 and 1⇔4 by Proposition 5.3 and Definition 5.2. That 2⇒3 is clear; that 3⇒1 follows by the corresponding argument of Theorem 4.5. $\square$

Thus, once completeness of a demand type is known, the weaker checks of Theorem 4.4 are sufficient to provide their 'complete' counterparts given here. Moreover, as is intuitive, completeness is preserved under unimodular basis changes:

Proposition 5.6. *Suppose G is a unimodular $n \times n$ matrix.*

1. *If $u : A \to \mathbb{R}$ is concave and $\mathcal{D}$-complete, then G^*u is concave and is $G^{-1}\mathcal{D}$-complete.*

2. *If $\mathcal{D}$ is complete then $G^{-1}\mathcal{D}$ is complete.*

Proof. See Appendix A.2. $\square$

The question remains of which demand types are complete. In Section 5.2 we develop a sufficient condition: unimodularity. However, this condition is not necessary. Here we give two examples, one unimodular and one not.

Example 5.7. The set $\mathcal{D}_{all}^n$ of *all* primitive integer n-vectors is complete. This is trivial: given any concave valuation and any prices $\mathbf{p}, \mathbf{p}'$ we can simply note the bundles demanded in improving steps on the straight line from $\mathbf{p}$ to $\mathbf{p}'$, interpolating additional primitive integer steps if necessary. It is of interest to emphasise this point, because many sub-types $\mathcal{D} \subsetneq \mathcal{D}_{all}^n$ are *not* complete. This fact stands in contrast to the property of unimodularity, which we shall come to in Section 5.2.

Example 5.8. The set $\mathcal{D}_a^n$ of the coordinate vectors is a complete demand type. As shown in Proposition 4.13, any concave valuation u of type $\mathcal{D}_a^n$ is additively separable. Thus, for any good i, the quantity demanded of any single good depends only on the price for that good. So if $\mathbf{x} \notin D_u(\mathbf{p})$ for any bundle $\mathbf{x}$ and price vector $\mathbf{p}$, then we may adjust the quantities of all goods independently, towards the desired levels. That is, we may make a series of $\mathcal{D}_a^n$-improvements given any starting bundle and price.

5.2 Unimodularity: a sufficient condition for completeness

We now introduce a condition on demand types which is sufficient for their completeness; strikingly, the same condition is also necessary and sufficient for existence of competitive equilibrium (see Section 6.3).

Throughout, we write "the *determinant* of vectors $\mathbf{w}^1, \ldots, \mathbf{w}^n$" to mean the determinant of the $n \times n$ matrix which has these vectors as its columns.[59] We say that a linearly independent set $\{\mathbf{w}^1, \ldots, \mathbf{w}^s\}$ of vectors is "an *integer basis* for the subset they span" if, whenever $\mathbf{y} \in \mathbb{Z}^n$ can be written as $\sum_{i=1}^s a_i \mathbf{w}^i$, in which $a_i \in \mathbb{R}$, then in fact $a_i \in \mathbb{Z}$ for $i = 1, \ldots, s$.

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[59]Changing the order of the vectors may change the sign of the determinant, so strictly speaking the determinant is a property of an ordered n-tuple of vectors. This detail does not concern us as we are only ever interested in the absolute values of determinants.
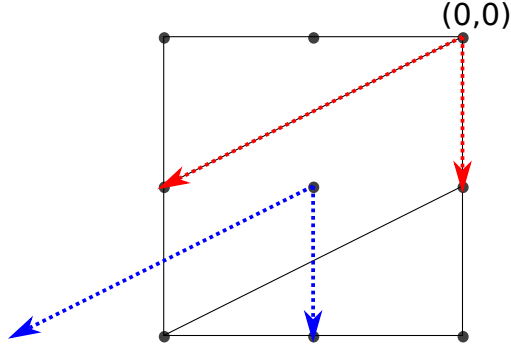
Figure 17: The SNP of Example 5.1, with the relevant vectors highlighted.

Definition 5.9. We say a demand type $\mathcal{D}$ is *unimodular* if any linearly independent set of vectors in $\mathcal{D}$ is an integer basis for the subspace they span.

It is clear that the set $\mathcal{D}_{all}^n$ of Example 5.7 does not have this property, whereas the set $\mathcal{D}_a^n$ of Example 5.8 does have it. We now clarify the implications of unimodularity by considering an example in which both unimodularity and completeness fail:

Example 5.1 revisited. Recall we considered a particular valuation u of demand type $\mathcal{D} = \{\pm(1,0), \pm(0,1), \pm(2,1)\}$. This demand type is not unimodular. The vector $(1,1)$ is in the span of $(0,1)$ and $(2,1)$ but it is not possible to express it as an integer combination of these vectors. This impossibility is illustrated in Figure 17, in which the vectors $(0,1)$ and $(2,1)$, starting from the origin, are highlighted in red.

It is similarly impossible to move on from the bundle $(1,1)$ to the bundle $(2,2)$ using this pair of directions, as highlighted in blue. Since these two vectors are the only two in $\mathcal{D}$ that change the quantity of Good 2 demanded, the failure of $\mathcal{D}$-completeness follows.[60]

To understand the relevance of unimodularity in general, consider a set of s linearly independent vectors from a demand type $\mathcal{D}$. These are the edges of an s-dimensional parallelepiped. This shape will contain no integer point (either in its boundary or in its interior) aside from its vertices, iff, our set of vectors is an integer basis for the subspace they span. Thus the scenario of Figure 17, in which the bundle $(1,1)$ gives difficulties, simply cannot arise if the demand type is unimodular.

This condition may appear difficult to check. However, returning to the parallelepiped, it is not too difficult to see that unimodularity holds iff the s-dimensional *volume* of the parallelepiped is 1. When $s = n$, this volume is simply the (absolute value of the) determinant of the vectors along its edges. When $s < n$, unimodularity is equivalent to the ability to add additional integer vectors that build the s-dimensional volume out into an n-dimensional volume of 1, i.e., so that the determinant is again ± 1. (In the case of Figure 17, the determinant in question is 2).

We gather these conditions with an alternative check for unimodularity when $s < n$, via appropriate determinants of submatrices, as follows:

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[60]If $(1,1)$ were in the demand type, then it would be possible to make a suitable step, confirming (as is already clear from Example 5.7) that unimodularity is not necessary.

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Remark 5.10. The following are equivalent, for a set of s linearly independent vectors in $\mathbb{Z}^n$:

1. they are an integer basis for the subspace they span;

2. A s-dimensional parallelepiped in $\mathbb{R}^n$ with vertices in $\mathbb{Z}^n$ and these vectors as edges contains no point in $\mathbb{Z}^n$ except its vertices;

3. they can be extended to a basis for $\mathbb{R}^n$, of integer vectors, with determinant ± 1;

4. among the determinants of all the $s \times s$ matrices consisting of s rows of the $n \times s$ matrix whose columns are these s vectors, the greatest common factor is 1.

Proofs of these facts may be found in Cassels (1959).[61] Note that when $s = n$ then both 2 and 3 remind us that the determinant of the vectors is ± 1. We refer to a set of vectors as unimodular if every linearly independent subset has these properties.

Our result is now:

Theorem 5.11. *If a concave demand type is unimodular, then it is complete.*

Proof. See Appendix A.2.[62] □

Using Remark 5.10.3 we can state a slightly weaker result that is more intuitive and very easy to check:

Corollary 5.12. *With n goods, a concave demand type $\mathcal{D} = \{\mathbf{v}^1, \ldots, \mathbf{v}^r\}$, in which $\mathbf{v}^1, \ldots, \mathbf{v}^r$ span $\mathbb{R}^n$, is complete if every subset of n vectors from $\mathcal{D}$ has determinant 0 or ± 1.*

The intuition for Theorem 5.11 is as described above regarding Figure 17. More detail is required because not every SNP-face is a parallelepiped, and general SNP faces may contain non-vertex points, even for a unimodular demand type. However, we show that, when unimodularity holds, we may step from a non-vertex point to another point in the SNP face, in the direction of a vector in $\mathcal{D}$. The additional assumption of concavity is required to ensure that this new point in the SNP face is indeed demanded at relevant prices; see Example A.3 for an example of what happens when it fails.

We will see below that a very nice example of a concave unimodular demand type is 'strong substitutes'. The relationship between this and the corresponding 'single' improvement property is well known (Gul and Stacchetti, 1999; Milgrom and Strulovici,

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[61]$1 \Leftrightarrow 3$ follows from Cassels (1959) Lemma I.1 and Corollary I.3. $1 \Leftrightarrow 4$ is Cassels (1959) Lemma I.2. For $1 \Leftrightarrow 2$ consider a parallelepiped P whose vertices are $\mathbf{y} + \sum_{i=1}^{s} a_i \mathbf{w}^i$ for $a_i \in \{0, 1\}$. If $\mathbf{z}$ is a non-vertex integer point in P, then $\mathbf{z} - \mathbf{y}$ exhibits the failure of 1. Conversely, if failure of 1 is exhibited by an integer $\sum_{i=1}^{s} b_i \mathbf{w}^i$ where b_i are not all integers, then $\mathbf{y} + \sum_{i=1}^{s} a_i \mathbf{w}^i$ exhibits failure of 2, where a_i is the non-integer part of b_i in each case.

[62]As shown there, in fact we prove a stronger result: we need only assume that, for any vectors $\mathcal{V} \subseteq \mathcal{D}$, there exist linearly independent vectors $\mathbf{w}^1, \ldots, \mathbf{w}^s \in \mathcal{D}$, whose span over $\mathbb{R}$ coincides with the span over $\mathbb{R}$ of $\mathcal{V}$ and such that, for any $\mathbf{v}^1, \ldots, \mathbf{v}^{s-1} \in \mathcal{V}$ and any $i = 1, \ldots, s$, the vectors $\mathbf{v}^1, \ldots, \mathbf{v}^{s-1}, \mathbf{w}^i$ are either linearly dependent or are a unimodular set. This property is clearly implied by unimodularity of $\mathcal{D}$, as we may take the set $\mathbf{w}^1 \ldots, \mathbf{w}^s$ to be any linearly independent and spanning subset of $\mathcal{V}$. However, it is not sufficient for unimodularity; the set $\mathcal{D} = \pm\{0, 1\}^3$ is not unimodular but does have this property.

2009). Corollary 5.5 and Theorem 5.11 together strictly generalise this result by showing that the complete $\mathcal{D}$-improvement property is satisfied by all valuations of type $\mathcal{D}$ for any unimodular concave demand type $\mathcal{D}$.[63] It follows from Seymour (1980) that there are many such demand types that are *not* a unimodular basis change of strong substitutes.

5.3 Unimodular Demand Types for Substitute Goods

Following from Theorem 5.11 and the arguments of Section 4.3.1 it is clear that a valuation will satisfy the following definition of substitutes if it is $\mathcal{D}_{os}^n$-complete:

Proposition 5.13. *If a concave valuation u is $\mathcal{D}_{os}^n$-complete then, for any prices $\mathbf{p}' \geq \mathbf{p}$ and any $\mathbf{x} \in D_u(\mathbf{p})$, there exists $\mathbf{x}' \in D_u(\mathbf{p}')$ such that $x_k' \geq x_k$ for all k such that $p_k' = p_k$.*

Proof. One may make use of $\mathcal{D}_{os}^n$-steps exactly as in Proposition 4.8, but now starting at any price $\mathbf{p}$. □

Note that the latter condition is provided as the definition of indivisible substitutes by Kelso and Crawford (1982).[64]

Following Theorem 5.11, then, we see:

Proposition 5.14. *If $\mathcal{D} \subsetneq \mathcal{D}_{os}^n$ is unimodular then every concave valuation of type $\mathcal{D}$ is a $\mathcal{D}_{os}^n$-complete valuation (and so satisfies the Kelso-Crawford definition).*

On the other hand, for *ordinary* substitutes the $\mathcal{D}_{os}^n$-complete property may fail: see Danilov et al. (2003, Example 6).[65] Note also that, in order to apply the proposition, the demand type with respect to which u is complete must itself be a subset of $\mathcal{D}_{os}^n$; since (as just noted) there are *ordinary* substitutes which are not $\mathcal{D}_{os}^n$-complete, and since every valuation is of the complete demand type $\mathcal{D}_{all}^n$ (see Example 5.7), there clearly exist ordinary substitute valuations that are $\mathcal{D}$-complete for some $\mathcal{D} \nsubseteq \mathcal{D}_{os}^n$, but are *not* $\mathcal{D}_{os}^n$-complete.

We refer to unimodular demand types $\mathcal{D} \subsetneq \mathcal{D}_{os}^n$ as 'unimodular substitute' demand types.

One example of a 'unimodular substitute' demand type is $\{\pm(-1, 1), \pm(-1, 2), \pm(0, 1)\}$. Note that this demand type does not contain both distinct coordinate vectors – an agent *cannot* be indifferent about whether to buy or sell an additional unit of good 1, while maintaining constant demand for good 2. But this is still a natural model if, for example, agents are manufacturers, selling good 1, which they can manufacture from different

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[63]It is also observed by Danilov et al. (2008, 2013) that 'interval concave functions' can equivalently be characterised by 'interval package improvements' – see our Example 5.24. However, as demonstrated in Example 5.24, the interval package demand type is a unimodular basis change of the strong substitute demand type, and so their result follows immediately from the Gul and Stacchetti result.

[64]There are various nomenclatures in use for this concept; Danilov et al. (2003) call them 'gross substitutes' following Kelso and Crawford (who strictly speaking deal only with the $\{0,1\}^n$ case); Milgrom and Strulovici (2009) call them 'weak substitutes'.

[65]Thus, Ausubel and Milgrom's (2002) definition (our Definition 4.6) is *not* identical to that of Kelso and Crawford (1982) when there are multiple units of three or more goods. The definitions *are* equivalent in the simpler cases $n = 2$ (see Baldwin, Klemperer and Milgrom, in preparation) and $A = \{0,1\}^n$ (Danilov et al. 2003, Corollary 5; see also Hatfield et al. 2011 Theorem A.1). See Baldwin, Klemperer and Milgrom (in preparation) for further discussion.

quantities of good 2, depending perhaps on the technology in use–indeed this is the natural generalisation of Hatfield et al. (2013) to multiple units of goods, see Examples 2.12 and 2.15.

The best-known example of a unimodular set of vectors also gives rise to a substitute demand type:

Definition 5.15. $\mathcal{D}_{ss}^n$ consists of those vectors in $\mathbb{Z}^n$ with at most one $+1$ and at most one -1 coordinate entry, and all others zero.[66]

Checking that this is a unimodular demand type is an application of a well-known result:

Theorem 5.16. $\mathcal{D}_{ss}^n$ *is a unimodular demand type, and is* maximal: *if* $\mathcal{D}_{ss}^n \subsetneq \mathcal{D}$ *then* $\mathcal{D}$ *is not unimodular.*

Proof. That the set of vectors that $\mathcal{D}_{ss}^n$ comprises is unimodular was first shown by Poincaré (1900); an attractive inductive proof was provided by Veblen and Franklin (1921), which we reproduce in Appendix A.2. Its maximality is easy to show. If a vector $\mathbf{w}$ were introduced with $|w_j| > 1$ for some j then the determinant of $\mathbf{w}$ with all $\mathbf{e}^i$ such that $i \neq j$ is w_j, contradicting unimodularity by assumption. If a vector $\mathbf{w}$ has two $+1$ coordinate entries, say in i and j, then the determinant of it with $\mathbf{e}^i - \mathbf{e}^j$ and all $\mathbf{e}^k$ such that $k \neq i, j$, has absolute value 2. Thus no additional vectors may be introduced without contradicting unimodularity. $\qquad\square$

From the unimodularity of $\mathcal{D}_{ss}^n$ immediately follows:

Corollary 5.17. [67] *A concave ordinary substitute valuation with domain $A = \{0, 1\}^n$ is a $\mathcal{D}_{os}^n$-complete valuation.*

Proof. By Proposition 4.8 such a valuation is of demand type $\mathcal{D}_{os}^n$. If $A = \{0, 1\}^n$ then the only possible SNP edges in $\mathcal{D}_{os}^n$, and hence the only possible facet normals, are those in $\mathcal{D}_{ss}^n$. $\qquad\square$

The vectors in $\mathcal{D}_{ss}^n$ were first related to substitutes by Tomizawa (1983), but without proof; proofs were offered by Danilov et al. (2003), and Fujishige and Yang (2003). An equivalent formulation was given by Milgrom and Strulovici (2009), who also show that there are several additional equivalent characterisations. Here we gather several relevant definitions:

Definition 5.18.

1. A valuation u is a *step-wise gross substitute valuation* if for any $\mathbf{p} \in \mathbb{R}^n$, any $\mathbf{x} \in D_u(\mathbf{p})$ and any $i \in \{0, \ldots, n\}$, either $\mathbf{x} \in D_u(\mathbf{p} + \epsilon\mathbf{e}^i)$ for all $\epsilon \geq 0$ or there exists $\epsilon \geq 0$ and $\mathbf{x}' \in D_u(\mathbf{p} + \epsilon\mathbf{e}^i)$ such that $x_i' = x_i - 1$ and $x_{-i}' \geq x_{-i}$.[68]

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[66]Danilov et al. (2003) say a valuation is a 'PM-function' (where PM stands for polymatroid) in this case.

[67]An analogous result is Danilov et al. (2003) Corollary 5.

[68]See Danilov et al. (2003).

2. A valuation u is a *strong substitute valuation* if, when we consider every unit of every good as a separate good, then they are ordinary substitutes.[69]

3. A valuation u satisfies the *ordinary (complete) law of aggregate demand* if for $k \in \{1, \ldots, n\}$, any $\epsilon > 0$ any $\mathbf{p} \in \mathbb{R}^n$ and any $\mathbf{x} \in \mathbb{Z}^n$ such that $\{\mathbf{x}\} = D_u(\mathbf{p})$ (resp. $\mathbf{x} \in D_u(\mathbf{p})$) there exists $\mathbf{x}' \in D_u(\mathbf{p} + \epsilon \mathbf{e}^k)$ such that $\sum_i x_i' \leq \sum_i x_i$.[70]

4. A valuation u satisfies the *consecutive integer property* if, for every $\mathbf{p} \in \mathbb{R}^n$ and every $i \in \{1, \ldots, n\}$, the set $\{x_i \mid \mathbf{x} \in D_u(\mathbf{p})\}$ consists of consecutive integers.[71]

Demand types provide nice ways to characterise the "law of aggregate demand" and the consecutive integer property, and these characterisations in turn make it much easier to demonstrate alternative characterisations of strong substitutes than is possible using traditional techniques. It is both intuitively clear and straightforward that (details are in Appendix A.2):

Lemma 5.19.

1. *A valuation u satisfies the ordinary (complete) law of aggregate demand iff it is of some (unimodular concave) demand type $\mathcal{D}$ such that, for all $\mathbf{v} \in \mathcal{D}$, either $\mathbf{v} \geq \mathbf{0}$, $-\mathbf{v} \geq \mathbf{0}$, or $\sum_{i=1}^n v_i = 0$.*

2. *A concave valuation u satisfies the consecutive integer property iff it is of some concave demand type $\mathcal{D} \subseteq \{-1, 0, 1\}^n$.*

It is then easy to use this Lemma to show the following, which slightly generalises Milgrom and Strulovici (2009, Theorem 13):[72]

Corollary 5.20. *For a valuation u, the following are equivalent:*

1. *u is of concave demand type $\mathcal{D}_{ss}^n$;*

2. *u is a strong substitute valuation;*

3. *u is a concave step-wise gross substitute valuation;*

4. *u is concave and satisfies the complete $\mathcal{D}_{ss}^n$-improvement property;*

5. *u is a concave ordinary substitute valuation and satisfies the ordinary law of aggregate demand;*

6. *u is a concave ordinary substitute valuation and satisfies the consecutive integer property;*

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[69]See Milgrom and Strulovici (2009). Note that, by Corollary 5.17, it is equivalent whether we define strong substitutes as all units of all goods being $\mathcal{D}_{os}^n$-complete substitutes, or just as all units of all goods being ordinary substitutes.

[70]See Hatfield and Milgrom (2005) for the 'complete' version.

[71]See Milgrom and Strulovici (2009).

[72]Milgrom and Strulovici (2009, Theorem 13) does not mention step-wise gross substitutes and assumes the $\mathcal{D}_{os}^n$-complete property in 5 and 6.

Proof. 1⇔4 is an application of Theorem 5.11. The complete $\mathcal{D}_{ss}^n$-improvement property is precisely the single-improvement property of Gul and Stacchetti (1999), so 4⇔2 follows from Milgrom and Strulovici (2009, Theorem 13).[73] That 1⇔3 is clear from Corollary 5.5.1⇔3 (and also given by Danilov et al. 2003, Proposition 7). That 1⇔5 follows since by Proposition 4.8 we know ordinary substitutes to be equivalent to demand type $\mathcal{D}_{os}^n$ and since by Lemma 5.19.1, for the ordinary law of aggregate demand to additionally hold, the only vectors in $\mathcal{D}_{os}^n$ we can allow are those in $\mathcal{D}_{ss}^n$. That 1⇔6 similarly follows from Lemma 5.19.2 since the vectors in $\mathcal{D}_{os}^n$ satisfying the consecutive integer property are precisely the vectors of $\mathcal{D}_{ss}^n$. □

We will in general refer to the valuations satisfying these equivalent conditions as 'strong substitutes' as this terminology appears to have become more widely used (and is briefer than 'concave step-wise gross substitutes'). We can now quickly and easily identify such valuations. It is immediate, for example, that the examples of Figures 1, 4a, 5, 13a, 13b, and 14a are all of type $\mathcal{D}_{ss}^2$, while our 3-dimensional example, Figure 6, has demand type $\mathcal{D}_{ss}^3$. However, Example 2.10 (Figure 3) has a facet with normal $(-1, 2)$ (the line segment between the prices $(4,3)$ and $(6,4)$), in addition to facets with normals $(1, 0), (0, 1)$, and $(1, -1)$, and so is not of type $\mathcal{D}_{ss}^2$, but *is* of type $\mathcal{D}_{os}^2$, as is the example of Figure 14b.[74]

Examples 2.14 and 2.15 revisited again. Recall we saw in Section 4.3.1 that the models of Kelso and Crawford (1982) and Hatfield et al. (2013) could be understood as ordinary substitute demand types. Now note that they are both also of demand type $\mathcal{D}_{ss}^n$. In the case of Kelso and Crawford (1982), all workers and all firms have valuations with domain contained in $\{0, 1\}^{m_1 + m_2}$. In the case of Hatfield et al. (2013), since each agent is restricted to *either* selling *or* buying any individual good, a re-ordering of goods shows their valuation has domain a subset of $\{0, 1\}^{n_1} \times \{-1, 0\}^{n_2}$ for some non-negative $n_1 + n_2 = n$.[75] Thus, following the same arguments as Corollary 5.17, every SNP edge, and so every facet normal, must be in $\mathcal{D}_{ss}^n$: the structure of preferences is again strong substitutes.

5.4 Unimodular Demand Types for Complementary Goods

Preferences for complementary goods are less thoroughly treated in the literature than preferences for substitutes, but we can use our techniques, and the analogy between the two cases, to develop some results for complements:

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[73]Strictly speaking, the proof of Milgrom and Strulovici (2009, Theorem 13) is incomplete, as the connection between the single-improvement property and the strong substitute property relies on their Theorem 2, whose proof is incomplete. However, Danilov et al. (2003, Corollary 5) provide the missing piece.

In the case that $n = 1$, the result that 1⇔2 follows from Kelso and Crawford (1982, Theorem 6).

[74]It can be shown that any strong substitutes preferences can be represented by a simple extension of the Bank of England's implementation of the Product-Mix Auction to allow negative bids. See Klemperer (2010, note 22) for the two-good case; Baldwin and Klemperer (in preparation) demonstrates this for the general case.

[75]Strictly speaking, a buyer might buy a good from one agent and sell a physically identical good to another agent in Hatfield et al. (2013) but because the transactions are independently priced we consider these goods to be distinct.

Proposition 5.21. *If a concave valuation is $\mathcal{D}_{oc}^n$-complete then, for any prices $\mathbf{p}' \geq \mathbf{p}$ and any $\mathbf{x} \in D_u(\mathbf{p})$ there exists $\mathbf{x}' \in D_u(\mathbf{p}')$ such that $x_k' \leq x_k$ for all k such that $p_k = p_k'$.*

Proof. One may make use of $\mathcal{D}_{oc}^n$-steps exactly as in Proposition 4.11, but now starting at any price $\mathbf{p}$. $\qquad\square$

Following Theorem 5.11 it is now easy to provide a sufficient condition for $\mathcal{D}_{oc}^n$-completeness:

Proposition 5.22. *If $\mathcal{D} \subsetneq \mathcal{D}_{oc}^n$ is unimodular then every concave valuation of type $\mathcal{D}$ is a $\mathcal{D}_{oc}^n$-complete valuation.*

One example of such a demand type is $\mathcal{D} = \{\pm(1,1), \pm(1,2)\}$. But, as with substitutes, our main interest is in unimodular demand types which are also subsets of $\{-1, 0, 1\}^n$. Indeed such demand types are of particular interest for complements, since they correspond to coalition-formation problems with transferable utility, though note that in that context, every valuation is automatically complete, since the domain of every individual is only two points.[76] By definition of $\mathcal{D}_{oc}^n$ the vectors in such demand types are either in $\{0,1\}^n$ or $\{-1,0\}^n$; we write $\mathcal{D} \subseteq \pm\{0,1\}^n$ for brevity. By Lemma 5.19, the consecutive integer property is also satisfied by valuations of such demand types; by Corollary 5.5.3 they are also "step-wise gross complements" valuations, where we define, in analogy with Danilov et al. (2003):

Definition 5.23. A valuation is a *step-wise gross complements valuation* if for any $\mathbf{p} \in \mathbb{R}^n$, any $\mathbf{x} \in D_u(\mathbf{p})$ and any $i \in \{0, \ldots, n\}$, either $\mathbf{x} \in D_u(\mathbf{p} + \epsilon\mathbf{e}^i)$ for all $\epsilon \geq 0$ or there exists $\epsilon \geq 0$ and $\mathbf{x}' \in D_u(\mathbf{p} + \epsilon\mathbf{e}^i)$ such that $x_i' = x_i - 1$ and $x_{-i}' \leq x_{-i}$.

For $n \geq 3$, however, there is *no* unique maximal unimodular demand type contained in $\pm\{0,1\}^n$.[77] We explain this by developing an example.

Example 5.24 ('Interval package' valuations, see Danilov et al. 2008, 2013)**.** Let G be the upper triangular matrix of 1s:

$$
G := \begin{pmatrix} 1 & 1 & \cdots & 1 \\ 0 & 1 & \cdots & 1 \\ \vdots & & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \end{pmatrix}.
$$

Then (following the notation of Section 4.4) the demand type $G\mathcal{D}_{ss}^n$ is unimodular. The vectors in $G\mathcal{D}_{ss}^n$ are of the forms $G\mathbf{e}^i = \sum_{k=1}^i \mathbf{e}^k$ and $G(\mathbf{e}^i - \mathbf{e}^j) = \sum_{k=j+1}^i \mathbf{e}^k$ for $i > j$ (as well as the negations of these). These are all therefore in $\pm\{0,1\}^n$, so $G\mathcal{D}_{ss}^n$ is a unimodular complements demand type.

Moreover, $G\mathcal{D}_{ss}^n$ has attractive economic properties. If the goods have a natural fixed order, then under $G\mathcal{D}_{ss}^n$, any contiguous collection of goods may be considered

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[76]See Example 2.16.

[77]If $n = 1$ then any valuation is a $\mathcal{D}_{oc}^n$-complete valuation; if $n = 2$ then the unique maximal unimodular complements demand type is $\pm\{0,1\}^2$ itself. For $n \geq 3$ the set $\pm\{0,1\}^n$ is not unimodular; see Example 6.13.

as complements by any agent. Such valuations may arise when agents consider, for example, bands of radio spectrum, or 'lots' of sea bed which might be developed for offshore wind (see Ausubel and Cramton, 2011).

Now, since the demand type $\mathcal{D}_{ss}^n$ is a maximal unimodular demand type (see Theorem 5.16) it follows that $G\mathcal{D}_{ss}^n$ is also maximal as a unimodular demand type.

However, it is easy to see that if $n \geq 3$ there exist vectors $\mathbf{v} \in \{0,1\}^n$, $\mathbf{v} \notin G\mathcal{D}_{ss}^n$ which do lie in *some* unimodular complements demand type. If we simply change the order of the goods, this corresponds to a unimodular basis change P, and $PG\mathcal{D}_{ss}^n$ has the same properties as $G\mathcal{D}_{ss}^n$. But if P is the permutation swapping coordinates 2 and n and leaving the rest unaltered, then $\mathbf{e}^1 + \mathbf{e}^n \in PG\mathcal{D}_{ss}^n$. However, $\mathbf{e}^1 + \mathbf{e}^n \notin G\mathcal{D}_{ss}^n$.

Moreover, there exist unimodular complements demand types that are not themselves unimodular basis changes of $\mathcal{D}_{ss}^n$; as sets of vectors, these were characterised by Seymour (1980). One such is as follows:

Example 5.25. Consider the demand type defined by the matrix

$$
D := \begin{pmatrix}
1 & 0 & 0 & 1 & 0 & 0 & 1 & 1 & 0 \\
0 & 1 & 0 & 0 & 1 & 0 & 1 & 0 & 1 \\
0 & 0 & 1 & 0 & 0 & 1 & 0 & 1 & 1 \\
0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1
\end{pmatrix}.
$$

Here, the first three goods are of value on their own, but the fourth is not; there are pairwise complementarities between any one of the first three goods together with the fourth, and there are additional complementarities between any pair of the first three goods when the fourth is also present. We interpret this as a matching model (cf. Example 2.16) in Section 6.3.1.

Claim 5.26. *The demand type of Example 5.25 is unimodular, and is not a basis change from 4-dimensional strong substitutes $\mathcal{D}_{ss}^4$.*

Proof. One may easily show this demand type is unimodular.[78]

To show it is not a basis change from 4-D strong substitutes, $\mathcal{D}_{ss}^4$, assume (for contradiction) there exists a unimodular matrix G such that $G^{-1}D$ consists entirely of distinct column vectors from $\mathcal{D}_{ss}^4$.[79] Since D has 9 columns, $G^{-1}D$ must include all but one of the 10 distinct vectors in $\mathcal{D}_{ss}^4$. Let $\mathbf{w} := (1,1,1,1,1,1,-1,-1,-1)$ and note that $D\mathbf{w}' = \mathbf{0}$, so $G^{-1}D\mathbf{w}' = \mathbf{0}$ also. It follows that every row $\mathbf{r}$ of $G^{-1}D$ satisfies $\mathbf{r}.\mathbf{w} = 0$. But there are precisely four vectors in $\mathcal{D}_{ss}^4$ with non-zero entry in any coordinate i ($\mathbf{e}^i$, and $\mathbf{e}^i - \mathbf{e}^j$ for the three values of $j \neq i$), so there are four non-zero entries in every row of the matrix whose columns are the 10 distinct vectors of $\mathcal{D}_{ss}^4$, and if we delete any one column, then at least one row must have exactly three non-zero entries. Since these three entries are ± 1, there is no way to add or subtract the three together to obtain zero; it is impossible that this row has zero dot product with $\mathbf{w}$. Thus no nine vectors of $\mathcal{D}_{ss}^4$ can form the columns of $G^{-1}D$, for any unimodular matrix G. $\square$

Moreover, it follows from the mathematical results of Grishukhin et al. (2010) that *all* unimodular demand types are a unimodular basis change from a demand type $\mathcal{D} \subseteq$

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[78]For example, it can be confirmed using Matlab that the determinant of every set of four columns of D is ± 1 or 0.

[79]Vectors which are the negation of one another are *not* considered "distinct" in this context.

$\pm\{0,1\}^n$; unimodularity is preserved under unimodular basis change, and so we obtain the surprising result:

Theorem 5.27. *Every unimodular demand type is a unimodular basis change of a unimodular complements demand type contained in $\pm\{0,1\}^n$.*

In other work we plan to use their results to generate more examples of these demand types, such as our Example 5.25.

5.5 Generalised Gross Substitutes and Complements (cf. Sun and Yang, 2006)

We can extend Sun and Yang's (2006, see also 2009) definition of "gross substitutes and complements" to permit multiple units of goods.[80] First recall:

Definition 5.28 (Sun and Yang, 2006, Definition 2.1)**.** A valuation $u : \{0,1\}^{n_1+n_2} \to \mathbb{R}$ is a *gross substitutes and complements valuation (in the sense of Sun and Yang)* if, for any price $\mathbf{p}$ and any $\mathbf{p}' = \mathbf{p} + \delta\mathbf{e}^i$ where $\delta > 0$, and any $\mathbf{x} \in D_u(\mathbf{p})$: if $i \leq n_1$ then there exists $\mathbf{x}' \in D_u(\mathbf{p}')$ such that $x'_k \geq x_k$ for all $k \leq n_1$ such that $k \neq i$, and $x'_k \leq x_k$ for all $k > n_1$; and if $i > n_1$ then there exists $\mathbf{x}'' \in D_u(\mathbf{p}'')$ such that $x''_k \leq x_k$ for all $k \leq n_1$, and $x''_k \geq x_k$ for all $k > n_1$ such that $k \neq i$.

We will write I_{n_1,n_2} for the $(n_1+n_2) \times (n_1+n_2)$ matrix $I_{n_1,n_2} := \begin{pmatrix} -I_{n_1} & \mathbf{0} \\ \mathbf{0} & I_{n_2} \end{pmatrix}$ where I_{n_i} is the $n_i \times n_i$ identity matrix, $i = 1,2$. Recall from Proposition 4.14 that, if $A \subsetneq \mathbb{Z}^{n_1+n_2}$ then, for any $u : A \to \mathbb{R}$, we define the valuation $I^*_{n_1,n_2}u : I^{-1}_{n_1,n_2}A \to \mathbb{R}$ via $I^*_{n_1,n_2}u(\mathbf{y}) = u(I_{n_1,n_2}\mathbf{y})$ for all $\mathbf{y} \in I^{-1}_{n_1,n_2}A$. Now we define:

Definition 5.29 (Cf. Shioura and Yang, 2013, Definition 2)**.** Let $A \subsetneq \mathbb{Z}^n$ be finite, and let $u : A \to \mathbb{R}$ be a valuation. Goods are *generalised gross substitutes and complements (GGSC)* if the goods may be reordered such that, for some $n_1 + n_2 = n$, the valuation $I^*_{n_1,n_2}u$ is a strong substitute valuation.

The corresponding demand type we define is as follows:

Definition 5.30. $\mathcal{D}^{n_1,n_2}_{GGSC}$ is the following set of vectors in $\mathbb{Z}^{n_1+n_2}$

$$\{\mathbf{e}^i, \mathbf{e}^j, \mathbf{e}^i - \mathbf{e}^{i'}, \mathbf{e}^i + \mathbf{e}^j, \mathbf{e}^j - \mathbf{e}^{j'} \mid i, i' \in \{1, \ldots, n_1\}, j, j' \in \{n_1 + 1, \ldots, n_1 + n_2\}\}.$$

It follows straightforwardly that:

Proposition 5.31. *A valuation is a GGSC valuation iff the goods may be reordered such that, for some $n_1 + n_2 = n$, it is of concave type $\mathcal{D}^{n_1,n_2}_{GGSC}$. If the domain of the valuation is $\{0,1\}^n$ then this holds iff it is a gross substitutes and complements valuation in the sense of Sun and Yang.*

---

[80]Shioura and Yang (2013) have independently made the same generalisation; their Theorem 2 shows, as we do below, that (in our language) "generalised gross substitutes and complements" (GGSC) satisfy the complete D^{n_1,n_2}_{GGSC}-improvement property.

Proof. It is not hard to see that $\mathcal{D}_{GGSC}^{n_1,n_2} = I_{n_1,n_2}^{-1} \mathcal{D}_{ss}^n$. So the first result follows by Definition 5.29 and by Proposition 4.14. The second result is provided by Sun and Yang (2006, Section 3), who show that Definitions 5.28 and 5.29 coincide when $A = \{0,1\}^n$. $\square$

As we know $\mathcal{D}_{ss}^n$ to be unimodular, it follows that $\mathcal{D}_{GGSC}^{n_1,n_2}$ is another unimodular demand type, and therefore (Theorem 5.11) it is also another complete demand type.

6 Aggregate Demand and Equilibrium

We now consider aggregate demand across many agents. In particular, we precisely identify the demand types for which competitive equilibrium always exists.

6.1 The structure of aggregate demand

First, we show that aggregate demand among agents may be understood in the way developed by Koopmans (1951): the aggregate value represents the most efficient use of the bundles available, in terms of generated welfare.

We have a finite set J of agents. Each agent j has a valuation u^j of integer bundles in a finite set A_j, so the bundles of interest on aggregate are $A := \{\sum_{j \in J} \mathbf{x}^j \mid \mathbf{x}^j \in A_j\}$, which we shall refer to as the *domain* of the aggregate valuation.[81] The aggregate demand at any price $\mathbf{p}$ is simply

$$D_{\{u^j\}}(\mathbf{p}) := \left\{ \sum_{j \in J} \mathbf{x}^j \mid \mathbf{x}^j \in D_{u^j}(\mathbf{p}) \right\}. \tag{4}$$

One way to find aggregate demand is to start with the valuation functions $u^j(\cdot)$, combine them to give an 'aggregate valuation function', and then proceed in exactly the same way as for individual demand. It is standard (see Appendix A.3) that if agents' preferences are quasilinear then one attains an aggregate valuation function $U : A \to \mathbb{R}$ as the greatest sum of valuations that can be attained by dividing any bundle $\mathbf{y} \in A$ between the agents, that is, the most efficient division of this bundle:

$$U(\mathbf{y}) := \max \left\{ \sum_{j \in J} u^j(\mathbf{x}^j) \mid \mathbf{x}^j \in A_j, \sum_{j \in J} \mathbf{x}^j = \mathbf{y} \right\}.$$

Now:

Proposition 6.1. $D_{\{u^j\}}(\mathbf{p}) = D_U(\mathbf{p})$ *for all* $\mathbf{p} \in \mathbb{R}^n$.

So we henceforth refer to $D_{\{u^j\}}(\mathbf{p})$ using the simpler notation $D_U(\mathbf{p})$.

However, the problem with this approach is that $U(\cdot)$ is very hard to work with–to find any value of $U(\mathbf{y})$, we need to consider all possible partitions of y among the agents, which is both time-consuming and unintuitive.

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[81]We could alternatively consider each agent as having a valuation over the full domain of the aggregate valuation A by letting $u^j(\mathbf{x}) := \max\{u(\mathbf{y}) \mid \mathbf{y} \in A_j, y_i \leq x_i, i = 1, \ldots, n\}$ for any $\mathbf{x} \in A$ for which this set is non-empty, and $u^j(\mathbf{x}) = -\infty$ otherwise.

It is straightforward, on the other hand, to start with the individual THs, $\mathcal{T}_{u^j}$, combine them to form an aggregate TH, $\mathcal{T}_{\{u^j\}}$, and find information about aggregate demand from that. Recall that the underlying set of $\mathcal{T}_{u^j}$ is those prices at which demand u^j is non-unique. So, since aggregate demand $D_U(\mathbf{p})$ is unique iff all individual demands $D_{u^i}(\mathbf{p})$ are, the underlying set of $\mathcal{T}_{\{u^j\}}$ is just the union of all the $\mathcal{T}_{u^j}$. Figure 18 illustrates this for the aggregate of the two agents' demands in our simple substitutes and complements example, Example 2.11.

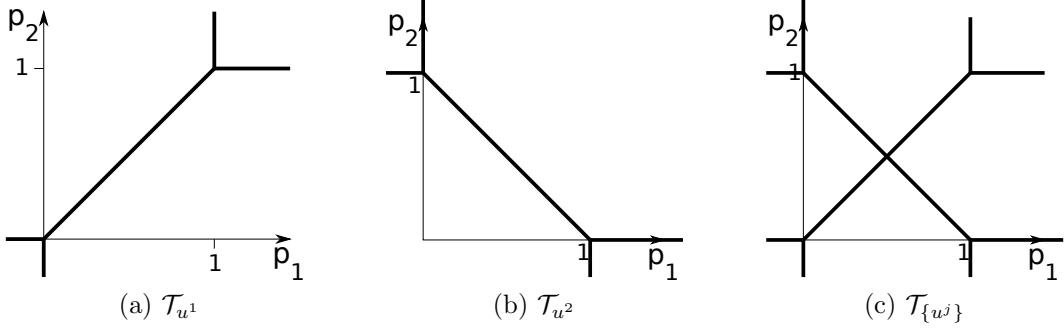(a) $\mathcal{T}_{u^1}$ (b) $\mathcal{T}_{u^2}$ (c) $\mathcal{T}_{\{u^j\}}$

Figure 18: (a) and (b) the THs of the individual demands of Example 2.11; (c) the TH of the aggregate of the two demands of Example 2.11.

$\mathcal{T}_{\{u^j\}}$ inherits the structure of a proper rational polyhedral complex from the individual THs, although the cells will not in general be exactly the same: if cell interiors from two different agents intersect, the cells are split up into new, smaller cells in $\mathcal{T}_{\{u^j\}}$ with a new, lower-dimensional, cell at their intersection. For example, in Figure 18c, the point $(\frac{1}{2}, \frac{1}{2})$ is a 0-cell, on the boundary of four *distinct* 1-cells.

It is easy to see that $\mathcal{T}_{\{u^j\}}$ also inherits a balanced weighting from the weightings of the individual THs. For any facet F of $\mathcal{T}_{\{u^j\}}$, let its weighting $w_{\{u^j\}}(F)$ be $\sum_{j \in J} w_j(F)$, in which $w_j(F)$ is the weight of the facet $F_j \supseteq F$ of $\mathcal{T}_{u^j}$, or $w_j(F) = 0$ if no facet $F_j \supseteq F$ of $\mathcal{T}_{u^j}$ exists. Since each individual TH is balanced, adding weightings in this way creates a balanced weighting.[82] And the change in aggregate demand as we cross a facet is just the sum of changes in individual demand.

So, since the underlying sets of $\mathcal{T}_{\{u^j\}}$ and $\mathcal{T}_U$ are the same, and so are their weightings, it follows (see Appendix A.3) that *as THs*,

Proposition 6.2. $\mathcal{T}_{\{u^j\}} = \mathcal{T}_U$.

So we will henceforth also refer to the aggregate TH, $\mathcal{T}_{\{u^j\}}$, using the simpler notation $\mathcal{T}_U$.

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[82]In more detail: let G be a $(n-2)$-cell in $\mathcal{T}_{\{u^j\}}$, let $F_1, \ldots, F_l$ be the facets adjacent to G, and let $\mathbf{v}_{F_k}$ be primitive integer vectors for each, chosen according to a coherent orientation. Then for every agent j, the equation $\sum_{k=1}^{l} w_j(F_k)\mathbf{v}_{F_k} = 0$ holds: if G is contained in an $(n-2)$-cell of $\mathcal{T}_{u^j}$ then, this follows from $\mathcal{T}_{u^j}$ being balanced; if G is contained only in a single facet of $\mathcal{T}_{u^j}$ then the only non-zero terms in this sum are those which first add and then subtract the weight of this facet to j; if $G \cap \mathcal{T}_{u^j} = \emptyset$ then the expression is identically zero. We conclude $\sum_{k=1}^{l} w_{\{u^j\}}(F_k)\mathbf{v}_{F_k} = \sum_{j \in J} \sum_{k=1}^{l} w_j(F_k)\mathbf{v}_{F_k} = 0$. Alternatively, one can see this by appealing to Appendix A.3, which confirms that the weightings are the same as those on $\mathcal{T}_U$ – being, of course, automatically balanced since it is the TH corresponding to $U(\cdot)$.

Thus simply "adding" the individual THs yields the aggregate TH.[83] If we know what is demanded in one UDR then, as before, we immediately know what is demanded in all the UDRs, without needing to directly consider the function U. And it is immediate that demand 'type' is preserved under aggregation:

Corollary 6.3. *Valuations u^j are of demand type $\mathcal{D}$ for all $j \in J$ iff the aggregate demand $\mathcal{T}_U$ is of demand type $\mathcal{D}$.*

Proof. This is immediate from Proposition 6.2 and the definition of $\mathcal{T}_{\{u^j\}}$. $\square$

6.2 Competitive Equilibrium and Stable Matchings

It is *not* the case that concavity of each individual demand implies concavity of the aggregate demand. (We will exhibit a simple example of this failure in Example 6.12.) And we have seen (Lemma 2.5) that if the function U is not concave, then there exists a bundle in A that is never demanded.

Of course, if there is a bundle which is not the aggregate demand of the agents for any price, then a competitive equilibrium does not exist when this is the bundle of goods available in the economy.

We cannot generally infer from only the aggregate TH whether there is a bundle that is never the aggregate demand–recall that the geometric construction does not tell us the precise demand set, $D_U(\mathbf{p})$, at all prices $\mathbf{p} \in \mathcal{T}_{\{u^j\}}$, so it is ambiguous from the geometry whether any integer vectors in Conv $D_U(\mathbf{p})$ that are not vertices of Conv $D_U(\mathbf{p})$ are in $D_U(\mathbf{p})$ (see Corollary 3.4). However, as we now show, we can start to answer these questions if we know not only the aggregate TH but also each individual TH.

The next subsection therefore provides conditions which guarantee that a competitive equilibrium *always* exists, by providing conditions which guarantee that the aggregate valuation $U(\cdot)$ is concave (without needing to explicitly calculate $U(\cdot)$). In particular, we are interested in the existence of equilibrium for agents with specified demand types, as defined in Section 4:

Definition 6.4. A (concave) demand type $\mathcal{D}$ *always has a competitive equilibrium* if, for every set of agents with (concave) demands of type $\mathcal{D}$, and for an economy endowed with any bundle in the domain of the aggregate valuation, a competitive equilibrium exists.

Note that since the demand of a single agent with non-concave valuation function fails to always have a competitive equilibrium, we are only interested in concave demand types here.[84]

A benefit of our method of categorising demand types is that it is straightforward that:

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[83]This is, of course, essentially the same point as the fact that in the Product-Mix Auction we can simply "add" individual bidders' sets of bids to form a single aggregate set of bids that represents bidders' aggregate demand. See Baldwin and Klemperer (in preparation) for further discussion.

[84]For any $\mathcal{D}$, there *always* exist *some* collection of agents with demands of type $\mathcal{D}$ which have a competitive equilibrium for any supply in their domain (e.g., consider a single agent with a concave valuation of type $\mathcal{D}$).

Proposition 6.5. *Always having a competitive equilibrium is a property that is preserved under unimodular basis changes.*

Proof. See Appendix A.3 □

Recall that in Example 2.16 we presented a model of matching with transferable utility in our framework, using 'coalition-agents' and 'person-goods'. The price paid for each person-good was precisely the net utility that person received if a matching took place. The typical question in such a matching model is whether a *stable* set of coalitions exists, i.e., a set such that no subset of people would prefer to deviate and form a new feasible coalition.

In this setting, the set of stable matchings corresponds to the set of core allocations, because if any group of people could make themselves better off by defecting and forming one or more new coalitions (with utility being transferable between people within, but not across, new coalitions) then a subset could defect and make a single coalition better off.[85] (It is also straightforward that this set also corresponds precisely to the set of core allocations of such a game when utility is also transferable across coalitions.[86]):

Definition 6.6 (cf. Gale and Shapley, 1962, and Shapley and Shubik, 1971)**.** In a model of matching with transferable utility within matches, a *stable matching* is an allocation in the *core* of the game among the people, that is, an assignment of each person to exactly one coalition and a set of transfers between the people within each coalition, such that there exists no feasible coalition which is not formed but whose formation would give strictly greater utility to all those people it would comprise.

So we have:

Theorem 6.7. [87] *A stable matching exists in a model of coalition formation with transferable utility iff there exists a competitive equilibrium in the re-formulation of Example 2.16.*

Proof. If a stable matching exists, the net utility received by each person is the price of that person (good). Each coalition which has formed must correspond to a coalition-agent with non-negative net utility: these do demand their corresponding people (goods) at these prices. Each coalition which has not formed cannot offer additional surplus to the people who would form it: the coalition-agent cannot afford the person-goods at the prevailing prices. So this is a competitive equilibrium.

---

[85]Since, in our setting, people are only ever part of a single group, questions about whether defectors remain parts of their previous groups do not arise. (Such questions are addressed in the literature on "group stability" – see Roth et al., 1992, Konishi and Ünver, 2006, etc. – and "setwise stability" – Sotomayor, 1999, Echenique and Oviedo, 2006, Klaus and Walzl, 2009, etc. – which impose additional requirement about which contracts are maintained after any deviation. See also Teytelboym, 2013.)

[86]If any group of people could make themselves better off by defecting and forming one or more new coalitions, perhaps *after* transfers between new coalitions, then at least one of the new coalitions is not receiving a positive transfer from the others, and that new coalition could have made itself better off by defecting on its own. So a stable matching must be in the core of the fully transferable utility game. (The converse is obvious.)

[87]Related literature includes Shapley and Shubik, 1971, Kaneko and Wooders, 1982, Kelso and Crawford, 1982, Chung, 2000, Eriksson and Karlander, 2001, Klaus and Nichifor, 2010, Talman and Yang, 2011, Chiappori et al., 2012, Hatfield et al., 2013, Hatfield and Kominers, 2013, etc.

Suppose a competitive equilibrium exists. A subset of people would only wish to deviate from their prescribed coalitions and form a new one together, if they could achieve a strictly higher net utility in the new coalition. The corresponding coalition-agent would have to be willing to offer a higher price for each of them. But the existence of such a coalition-agent would contradict competitive equilibrium. □

Recall that a person's 'price' is the minimum a coalition needs to pay to 'buy' him, in competitive equilibrium; any excess surplus can be split in any way among the people in a coalition.

6.3 When does Competitive Equilibrium exist?

We now state and explain a theorem which provides a necessary and sufficient condition for any set of valuations in a demand type to always have a competitive equilibrium.

This Theorem requires much weaker assumptions about agents' preferences than used in the existing leading economics literature, so our condition for equilibrium is correspondingly much more general. In particular (see Section 6.4.3) it is *not* necessary for all agents to have strong substitute demands (or some basis change thereof) for equilibrium to always exist.[88] Instead, concavity and the unimodularity condition of Section 5.2 are all that are required.

A remarkable series of papers by Danilov, Koshevoy and their coauthors, has developed results that are very closely related to ours. In particular, Theorems 3 and 4[89] of Danilov et al. (2001) together provide a *sufficient* condition for equilibrium ("$\mathcal{D}$-concavity" of valuations, where $\mathcal{D}$ is unimodular), which is analogous to our condition on demand types. However, the interpretation or usefulness of their result is not made clear; by contrast, the theorem we state both demonstrates the applicability of the result, and clarifies the connections to existing economic results.[90]

Danilov et al. also prove no *necessity* result. Because they have not developed their definition as a taxonomy of demand, in the way we do with demand types, they do not show the necessity of unimodularity of $\mathcal{D}$ for the existence of competitive equilibrium. Using demand types, however, a necessity result can easily be developed.[91]

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[88]For example, results such as those of Kelso and Crawford (1982), Hatfield and Kojima (2008), and Hatfield et al. (2013) are necessary 'in the maximal domain sense', in Hatfield et al. (2013)'s words. That is, in our language, they show that equilibrium always exists for some demand type $\mathcal{D}$, but that if one agent has preferences outside of $\mathcal{D}$ then this may fail.

[89]The proof of Theorem 4 is given by Danilov and Koshevoy (2004, Theorem 2).

[90]We will see that Theorem 6.8 unifies the results on competitive equilibrium in, for example, Kelso and Crawford (1982), Hatfield and Milgrom (2005), Sun and Yang (2006), Milgrom and Strulovici (2009), and Hatfield et al. (2013); these are all special cases of the theorem, and many other cases can be constructed. The absence in Danilov et al.'s work of the notion of demand types, and its presentation in relatively unfamiliar terms (namely the relationships between sets of primitive integer vectors which are parallel to edges of specific collections of integral pointed polyhedra and the "classes of discrete convexity" that they define) seems to have resulted in leading economists, and the leading related existing literature, being unaware of their work or its implications. (We were also unaware of their work until after we had developed our own results.)

[91]The sufficiency part of our theorem follows from combining Theorems 3 and 4 of Danilov et al. (2001). To understand the relationship between these theorems and our Theorem 6.8, observe that in their Theorem 4 certain sets of "primitive [integer] vectors, which are parallel to edges of" a certain "collection of integral pointed polyhedra" form a set $\mathcal{D}$ defining one of our demand types; furthermore,

Danilov et al. additionally state their results under different assumptions from ours. They assume the domain, A, of every agent's valuation is $\mathbb{Z}_{\geq 0}^n$, which precludes, for example, the application to Hatfield et al.'s (2013) model which our more general assumption permits.[92]

Finally, although the techniques we use to prove our results are novel, they seem simpler and more accessible to economists than Danilov et al.'s *very* advanced mathematical techniques. So we will prove our theorem using our alternative method, which understands the result as an application of "intersection multiplicities" in tropical geometry.[93]

Theorem 6.8. *A concave demand type $\mathcal{D}$ always has a competitive equilibrium iff it is unimodular.*

(Recall from Definition 6.4 that competitive equilibrium 'always exists' for a (concave) demand type iff, for any set of agents with (concave) valuation of that type, and any supply bundle in the convex hull of aggregate demand, there exists a price such that the market clears.)

As in Section 5.2, the intuition is that the volume of a SNP face which is a parallelepiped in $\mathbb{R}^n$, with vertices in $\mathbb{Z}^n$ and edges in $\mathcal{D}$, cannot exceed 1 if the set of vectors that form its edges is unimodular (see Remark 5.10), so if $\mathcal{D}$ is unimodular, such an SNP face of aggregate demand contains no integer points other than its vertices. So there is also no integer bundle "hidden" inside such an intersection of individual agents' THs, so no integer bundle that is never chosen at any price vector, and competitive equilibrium therefore always exists. On the other hand, if the set of edges exhibits failure of unimodularity, then such a parallelepiped's volume does exceed 1, there does exist a bundle not at a vertex, and–we will see–such bundles may not be chosen at any price vector, so competitive equilibrium may fail.

When $s = n$, the volume in question is simply the (absolute value of the) determinant of the vectors along its edges. So, as with Corollary 5.12, if the set of aggregate demands is in the same dimension as the number of goods, we can re-state the theorem in a form that is easier to check:

Corollary 6.9. *With n goods, a concave demand type $\mathcal{D} = \{\mathbf{v}^1, \ldots, \mathbf{v}^r\}$, in which $\mathbf{v}^1, \ldots, \mathbf{v}^r$ span $\mathbb{R}^n$, always has a competitive equilibrium iff every subset of n vectors from $\mathcal{D}$ has determinant 0 or ± 1.*

In the more general case of Theorem 6.8 we allow demand types $\mathcal{D}$ that ignore some directions of good availability. In such a $\mathcal{D}$ there are no collections of n linearly

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the "classes of discrete convexity" they define are analogous to a set of demand sets $D_u(\mathbf{p})$ such that $(\text{Conv } D_u(\mathbf{p})) \cap \mathbb{Z}^n = D_u(\mathbf{p})$ in every case and such that this property is preserved under aggregation. It is not hard to also show, using our Proposition 2.6, the necessity of a demand type being a "class of discrete convexity" for competitive equilibrium to always exist, and in this way we can also derive our *necessity* result from their work.

[92]In fact Danilov et al.'s assumption seems unnecessary for them, so we could develop our full theorem by extending their work. See our note 91, above. See also our discussion about the distinction between their approach and ours in the introduction to our Section 4. Their work also covers some of the examples in Section 6.4, as we note in that Section.

[93]It was this theory that inspired our (independent) development of our results. Full details of our proof are in Appendix A.3.

independent vectors, so every subset of n vectors has determinant 0, and the check of Corollary 6.9 tells us nothing.

In this case, however, we can use one of the equivalent conditions in Remark 5.10.3 and 5.10.4.

A demand type, $\mathcal{D}$, always has a competitive equilibrium iff all integer bundles (i.e., all lattice points) in any type-$\mathcal{D}$ SNP of aggregate demand are demanded for some price. It is immediate that any integer bundle that is at a vertex of the SNP is demanded (recall Corollary 3.4). On the other hand, any integer lattice point in the SNP of aggregate demand that is not a vertex is "hidden" inside the corresponding intersection of the individual agents' THs. Such a bundle is in the convex hull of the aggregate demands of the agents, at the price at which their THs intersect. Since Lemma 2.6 tells us that this is the only possible price at which such a bundle can be demanded, the question is, therefore, whether such a bundle is always demanded at the intersection price. So we can prove Theorem 6.8 by considering the forms such intersections may take.

We begin, in the next subsection, by considering a simple special case, which both proves the necessity of unimodularity for competitive equilibrium to always exist, and is of independent interest in understanding stable matching. The subsequent subsection extends this special case to prove the general case of the Theorem.

6.3.1 Competitive Equilibrium with "Simple" Intersections; and Stable Matchings

The simple case we start with is that for which an intersection between individual agents' THs lies in the interior of a facet of each agent, so each agent is indifferent between precisely two bundles, and the set of vectors normal to these facets is linearly independent. We show that, for such simple intersections, competitive equilibrium exists for every possible supply bundle if and only if this set of vectors is unimodular.[94] Using this result, examples of failure of equilibrium are always easy to construct.

Proposition 6.10. *Consider $s \leq n$ agents each of whose demand set includes precisely 2 bundles at price $\mathbf{p}$, i.e., $\#D_{u^i}(\mathbf{p}) = 2$, for $i = 1, \ldots, s$. Write $\mathbf{v}^i$ for the difference between the two bundles demanded by agent i (so $\mathbf{v}^i$ is normal to i's facet of demand at $\mathbf{p}$). Suppose the s vectors $\mathbf{v}^1, \ldots, \mathbf{v}^s$ are linearly independent. Write U for the aggregate valuation. There exists an integer bundle in $\mathrm{Conv}\, D_U(\mathbf{p})$ which is not demanded at any price iff vectors $\mathbf{v}^1, \ldots, \mathbf{v}^s$ do not form a unimodular set.*

Proof. By Lemma 2.6, an integer bundle in $\mathrm{Conv}\, D_U(\mathbf{p})$ is not demanded at any price iff it is not in $D_U(\mathbf{p})$. Now, each individual agent i's demand at $\mathbf{p}$ has the form $D_{u^i}(\mathbf{p}) = \{\mathbf{y}^i + \delta_i \mathbf{v}^i \mid \delta_i \in \{0, 1\}\}$, where $\mathbf{y}^i$ is the bundle demanded on the appropriate side of the TH facet. So the set of bundles demanded on aggregate at $\mathbf{p}$ is

$$D_U(\mathbf{p}) = \left\{ \mathbf{y} + \delta_1 \mathbf{v}^1 + \cdots + \delta_s \mathbf{v}^s \mid \delta_i \in \{0, 1\}; i = 1, \ldots, s \right\},$$

where $\mathbf{y} = \sum_i \mathbf{y}^i$. These points are precisely the vertices of an s-dimensional parallelepiped in $\mathbb{Z}^n$ (since its edges, the $\mathbf{v}^i$, are linearly independent). There exists an

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[94]Unimodularity is equivalent to the *tropical intersection multiplicity* being equal to one in such a case (see e.g. Osserman and Payne, 2013).

integer bundle in Conv $D_U(\mathbf{p})$ which is not in $D_U(\mathbf{p})$ iff this parallelepiped contains an integer bundle which is not a vertex, and, by Remark 5.10.1 and 2, this holds iff the set $\{\mathbf{v}^1, \ldots, \mathbf{v}^s\}$ is not unimodular. $\qquad \square$

This result tells us more than just the necessity of unimodularity for existence of competitive equilibrium. It shows us how to construct simple examples of failure of equilibrium: for any non-unimodular set of vectors, we simply need to choose a price vector, and then choose valuations so agents are all indifferent between exactly two bundles at this price vector.

In particular, *every* intersection of individual THs is of the form of Proposition 6.10 if every agent's domain of valuations contains only two bundles, as in the application to coalition-formation, Example 2.16, so Proposition 6.10 is of particular value in developing results in this context. For example, it is immediate that a stable matching always exists for *every* set of people (not necessarily just one person of each type) iff $\mathcal{D}$ is unimodular:

Corollary 6.11. *For every model of coalition formation with transferable utility of type $\mathcal{D}$, a stable matching exists for every set of people iff $\mathcal{D}$ is unimodular.*

Proof. Follows immediately from Theorem 6.7 and Proposition 6.10. $\qquad \square$

Example 5.25 revisited. Recall from Section 5.4 that the columns of the matrix

$$D := \begin{pmatrix} 1 & 0 & 0 & 1 & 0 & 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 \end{pmatrix}$$

define a unimodular demand type. Since, also, all its entries are 0 or +1, it is an example of a coalition-formation problem for which a stable match always exists. It might, for example, model the demand for three workers (the first three goods) and a manager (the fourth good). The first three columns of D show that each of the three workers has value on his own; the manager on her own is worthless (because $\mathbf{e}^4 \notin \mathcal{D}$), but the middle three columns of D show that the manager increases the value of any one of the workers, and the last three columns of D show that there are also complementarities between any two of the workers if (but only if) the manager is also present.

6.3.2 Proof of the Equilibrium Existence Theorem in the General Case

The *necessity* of unimodularity for competitive equilibrium is demonstrated immediately by considering intersections of individual agents' THs that take the simple form of Proposition 6.10.

We now prove in two stages that unimodularity and concavity are *sufficient* for competitive equilibrium to always exist.

First, we show in Proposition A.4 that all the integer bundles in the convex hull of the demands at any "nice" intersection of agents' THs are always demanded. Our definition of a "nice" intersection (see Proposition A.4) covers any generic intersection

at a single price. For example, in two dimensions, two lines crossing at a single point is "nice", but two coincident lines is not "nice", and nor is three lines crossing at a single point; in three dimensions, either three planes meeting in a single point, or a line meeting a plane in a single point is "nice".

The important property of "nice" intersections is that the changes in bundles considered by the different agents (as each agent crosses between different regions of its TH) are always linearly independent. This means that any change in the aggregate supply, which remains in the convex hull of aggregate demand at at this price point, can be straightforwardly and uniquely apportioned between the individual agents, by simply assigning to each individual agent that part of the aggregate change that follows its direction of change. Unimodularity of $\mathcal{D}$ implies that, if the aggregate change is by an integer bundle, then so are each of these individual changes. The concavity of each individual agent's valuation then means that each agent demands its new assigned bundle. So each separate component of the total bundle is demanded by an individual agent, and the aggregate bundle is therefore also demanded.

The second half of the proof of the sufficiency part of the theorem proceeds by showing that generically all TH intersections are "nice". That is, there always exist arbitrarily small perturbations of all agents' valuations that lead to a situation in which all intersections *are* nice, and so all bundles *are* demanded on aggregate at some price. But if, prior to these perturbations, there exists an integer bundle which is *not* demanded, then the aggregate value from this bundle must be a finite amount lower than the valuation that would be required for it to be demanded. So we can therefore make arbitrarily small perturbations in valuations that are on the one hand small enough that the integer bundle in question can still not be demanded at any price, but on the other hand mean that all bundles *are* demanded on aggregate at some price–a contradiction. We give the details in Appendix A.3.

6.4 Examples

6.4.1 Examples of non-existence of equilibrium

We first illustrate our result with two simple examples of non-existence of equilibrium; as in Section 5.2 we can see the importance of unimodularity by examining examples in which it fails. Proposition 6.10 shows us how to construct a failure of competitive equilibrium in such a case. We could mirror the demand type of Example 5.1 again, but instead we use the even simpler Example 2.11: substitutes and complements.

Example 6.12. [95] Suppose $\mathcal{D}$ can be represented by

$$D = \left(\begin{array}{cccc} 1 & 1 & 0 & 1 \\ -1 & 0 & 1 & 1 \end{array} \right)$$

in which the first three column vectors together yield the substitutes demand, and the last three column vectors together yield complements demand. Trivially, the matrix formed by the first and last column has determinant 2, so equilibrium need not exist.

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[95]See Danilov et al. (2001, Example 1) and Hatfield et al. (2013, Example 2).

Our Example 2.11 is of this type: we repeat its valuation functions for the "substitutes agent" and "complements agent" respectively, below:

$x_1 = 1$	$x_1 = 0$	u^1
1	0	$x_2 = 0$
1	1	$x_2 = 1$

and

$x_1 = 1$	$x_1 = 0$	u^2
0	0	$x_2 = 0$
1	0	$x_2 = 1$

Note that *both* these valuation functions are concave. However, the aggregate valuation function, which we give in Figure 19a is *not* concave, as can be easily seen by observing that $(U(1,0) + U(0,1) + U(2,1) + U(1,2))/4 > U(1,1)$. This inequality is also apparent in Figure 19b which shows a 3-dimensional illustration of U together with the face of $\widehat{A}$ (see equation (3) and Section 3.2) that corresponds to the price vector $(\frac{1}{2}, \frac{1}{2})$. It follows

$x_1 = 2$	$x_1 = 1$	$x_1 = 0$	U
1	1	0	$x_2 = 0$
2	1	1	$x_2 = 1$
2	2	1	$x_2 = 2$

(a) Aggregate valuation.

(b) 3 dimensional illustration of the aggregate valuation, showing the face of $\widehat{A}$ that corresponds to the price vector $(\frac{1}{2}, \frac{1}{2})$.
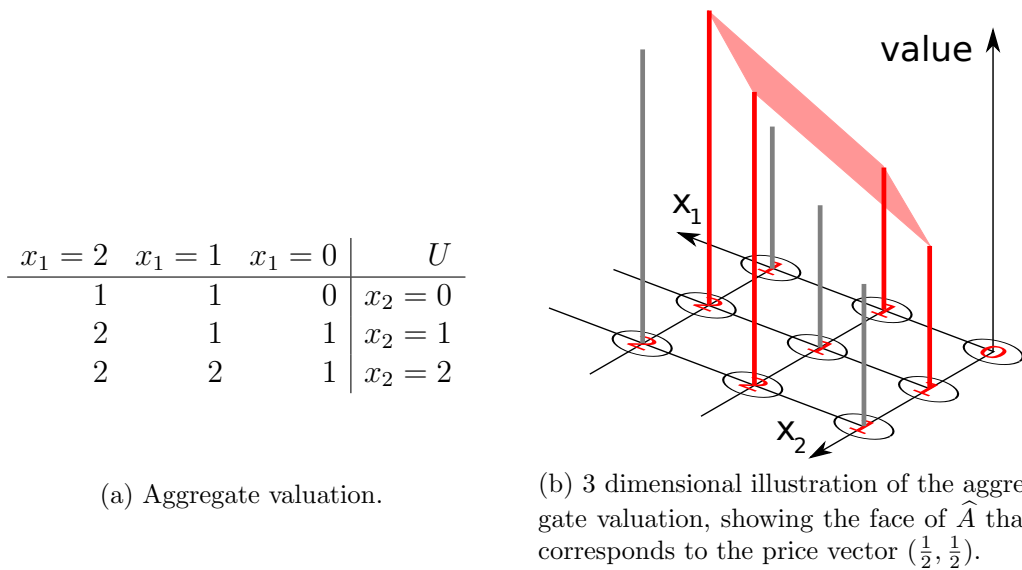
Figure 19: The aggregate valuation of Example 6.12.

that all the bundles $(1,0)$, $(0,1)$, $(2,1)$, and $(1,2)$ are demanded at this price, while the bundle $(1,1)$ is "hidden" at the intersection of the diagonals of the TH at the price, $(\frac{1}{2}, \frac{1}{2})$, and is never demanded at any price. So aggregate demand is never $x_1 = x_2 = 1$. The SNP and the TH of the individual and aggregate demands are shown in Figure 20. Observe in Figure 20c that in the aggregate SNP the bundle $(1,1)$ is not a vertex, and the area of the diamond is $\det \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix} = 2$.

Of course, our analysis only shows that equilibrium *may* not exist for this type of demand. Equilibrium would exist if, for example, the "complements" consumer had valuation 3 for the combination of 1 unit of each of x_1 and x_2. In that case the facets corresponding to the vectors $(1,1)$ and $(1,-1)$ would not intersect, so Proposition 6.10 does not apply. We will return to this issue in Section 6.5.

Example 6.13. [96] Consider a set of "complements" consumers each of whom is only interested in a different pair of goods. One context in which such a situation may arise

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[96]Sun and Yang (2011), and also Teytelboym (2014), have independently considered the demand described in this example, using alternative methods that extend Sun and Yang (2006), showing as we do that equilibrium always exists iff n is even. See also Footnote 98. If we use the 'matching'
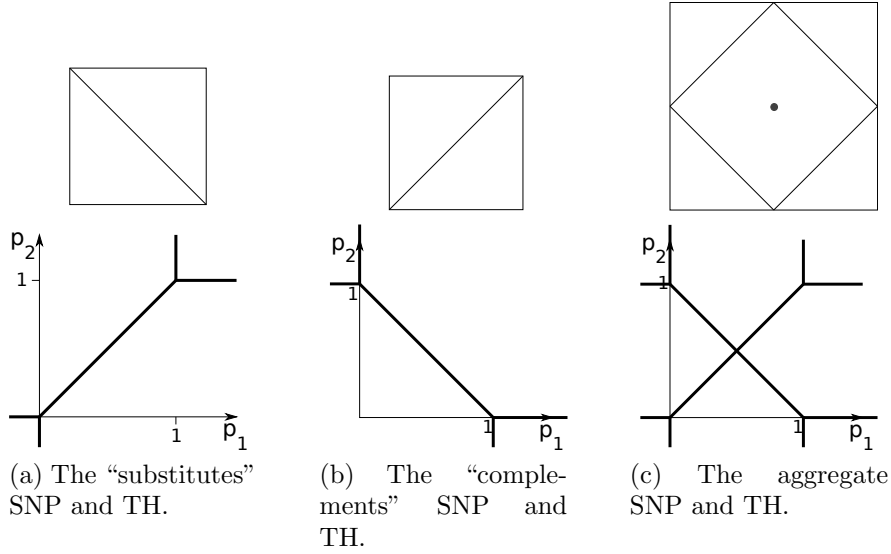
(a) The "substitutes" SNP and TH.

(b) The "comple-ments" SNP and TH.

(c) The aggregate SNP and TH.

Figure 20: The individual and aggregate SNPs and THs for Example 6.12.

is the "coalition formation" of Example 2.16–recall that in this case a stable matching is given by a competitive equilibrium (Theorem 6.7).

Moreover, assume that there is a cycle in the pairs of goods that these consumers wish for. That is, we can number both consumers and goods $1, \ldots, n$, such that every consumer $i < n$ demands goods i and $i + 1$, which it sees as perfect complements, and consumer n demands goods n and 1. It is not hard to see that:[97]

$$
\text{if } D = \begin{pmatrix} 1 & 0 & 0 & & 0 & 1 \\ 1 & 1 & 0 & & 0 & 0 \\ 0 & 1 & 1 & & 0 & 0 \\ \cdot & 0 & 1 & \cdot & & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & 0 & & \cdot & 1 & 0 \\ 0 & 0 & 0 & & & 1 & 1 \end{pmatrix} \text{ then } \det D = \begin{cases} 0 & \text{if } n \text{ is even} \\ 2 & \text{if } n \text{ is odd.} \end{cases}
$$

So if n is odd, there exist agents with demands of this type such that equilibrium does not exist.

Indeed, one can see directly that equilibrium fails in the simplest symmetric case: if each consumer has valuation 1 for any allocation that includes the pair it desires, and valuation 0 for any other allocation, aggregate demand is never exactly 1 unit of each good. To see this, note that at least one good, w.l.o.g. good 1, would not be part of a pair. So $p_1 = 0$. Therefore $p_2 \geq 1$ (else consumer 1 would demand the pair of goods 1 and 2). So $p_2 = 1$, and therefore $p_3 = 0$, since otherwise good 2 would not be demanded, and consumer 2 therefore buys goods 2 and 3. Therefore $p_4 \geq 1$ (else consumer 3 would demand goods 3 and 4). So $p_4 = 1$, and $p_5 = 0$, etc. In particular, $p_j = 0$ if j is odd. But in that case, consumer n wishes to buy goods n and 1, which is a contradiction.

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interpretation of Example 2.16 then this example is the transferable utility version of the 'no odd rings' condition of Chung (2000).

[97]To see this easily, expand by the first row: noting the "1"s in the first and the last column of that row, we have $\det D = 1(1) + (-1)^{n-1}(1)$.

On the other hand, if n is *even*, the columns of D are not linearly independent, but if we exclude the ith column, for any i, the remaining $n-1$ rows are linearly independent and can trivially be extended to n linearly independent vectors with determinant 1 by adding the column $\mathbf{e}^i$, so Theorem 6.8 then shows that equilibrium *always* exists. For example, in the simple symmetric case, $p_j = 0$ if j is odd and $p_j = 1$ if j is even, for all j, supports $q_i = 1$ for all i as an aggregate demand.

6.4.2 Strong substitutes and Generalised gross substitutes and complements

Recall from Section 5.3 that a valuation is 'strong substitutes', in the terminology introduced by Milgrom and Strulovici (2009), if every unit of every good is an ordinary substitute for every other unit of every good (including being an ordinary substitute for every other unit of the same good). We showed in Proposition 5.20.2 that strong substitutes are precisely are concave demand type $\mathcal{D}_{ss}^n$; the latter may be presented as $\{\mathbf{e}^i, \mathbf{e}^i - \mathbf{e}^j \mid i, j = 1, \ldots, n; i < j\}$ (see Section 5.3).

We know $\mathcal{D}_{ss}^n$ is unimodular (see Theorem 5.16), so another of the pleasing properties of 'strong substitutes' is that equilibrium always exists:

Proposition 6.14 (Milgrom and Strulovici, 2009, Theorem 19; cf. Danilov et al., 2003, Proposition 7 and Danilov et al., 2001, Example 4)**.** *Equilibrium always exists when agents' demands are strong substitutes.*

Recall that we showed that $\mathcal{D}_{ss}^n$ is not just unimodular, but is also maximal as a unimodular demand type (Theorem 5.16). This implies:

Proposition 6.15 (Gul and Stacchetti, 1999, Theorem 2, Milgrom and Strulovici, 2009, Theorem 16; Hatfield et al., 2013, Theorem 7)**.** *Given any one agent who does not have a strong substitute valuation, we can find strong substitute valuations for other agents such that competitive equilibrium fails to exist.*

Indeed, applying Proposition 6.10, we see that very simple valuations for the additional agents will suffice: each additional agent need only consider either whether or not to demand one unit of one good, or whether or not to swap one unit of one good for one unit of another.

Moreover, it is now trivial to reproduce:

Corollary 6.16 (Milgrom and Strulovici, 2009, Theorem 20; Danilov et al., 2003, Proposition 5)**.** *If u^j is a strong substitute valuation for all $j \in J$, then the aggregate valuation U is a strong substitute valuation.*

Proof. If u^j is of concave demand type $\mathcal{D}_{ss}^n$ for $j \in J$, then U is of type $\mathcal{D}_{ss}^n$ by Corollary 6.3. By Proposition 6.14 (and Lemma 2.5) U is also concave; applying Corollary 5.20 completes the proof. $\square$

Because equilibrium existence is preserved under unimodular basis changes (the clarity of this is one of the benefits of our representation of demand), an elementary application of Proposition 6.14 is:

Corollary 6.17 (cf. Sun and Yang, 2006, Theorem 3.1 and Shioura and Yang, 2013, Theorem 5.). *Equilibrium always exists for the 'generalised gross substitutes and complements' type of demand.*

Proof: Immediate from Propositions 6.14, 5.31 and 6.5. □

Note this corollary also provides another proof of Example 6.13's "even cycle of complements" result. If we separate the goods into two classes corresponding to the odd- and even-numbered goods, and re-order so that all the odd ones come first, demand is then of type $\mathcal{D}_{GGSC}^{n/2,n/2}$, so Corollary 6.17 applies.

Moreover, we can now generalise further to an even *more* general style of GGSC-like demand, in which goods are separated into an arbitrary number of groups, with goods within the same group being strong substitutes, but with 1-1 complementarities between some pairs of groups (that is, for those pairs of groups, each good in one of the groups may exhibit 1-1 complementarities with any good in the other group). If all the "cycles" formed by the sequences of "paired" groups are of even length, then we can again separate the groups of goods into two classes, so that the demand is again GGSC demand, and so always has a competitive equilibrium. But if any odd cycle exists then, just as in Example 6.13, competitive equilibrium may fail.[98]

6.4.3 When is Strong Substitutes a necessary condition for equilibrium?

Danilov and Grishukhin (1999) provided a characterisation of all of (what we call) unimodular demand types, including a list giving, up to unimodular basis change, all maximal such types up to dimension 6. From this list it is immediate that

Theorem 6.18. *If $n \leq 3$, equilibrium always exists for a concave demand type if and only if it is a unimodular basis change from strong substitutes, or a subset thereof.*

With $n > 3$, there exist concave demand types for which equilibrium always exists, which are not *a unimodular basis change from strong substitutes, or a subset thereof.*

That is, while if there are at most three goods, all unimodular demand types are unimodular basis changes from strong substitutes, this is far from true more generally. Indeed, we already showed this in Example 5.25 which, moreover, has *only* complementary relationships. This example also provides (see Theorem 6.7) a concrete example of preferences for which a stable coalition structure always exists – although coalitions may consist of one, two or three agents.

Furthermore, recall (Theorem 5.27) that *every* unimodular demand type, that is, *every* demand type for which competitive equilibrium is guaranteed, *is* a unimodular basis change of a unimodular complements demand type–in stark contrast to conventional wisdom about the "necessity" of substitutes for competitive equilibrium.[99]

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[98]This result has independently been established by Sun and Yang (2011), and also Teytelboym (2014); the latter paper gives fuller details.

[99]For example, Gul and Stacchetti (1999, p. 96) state "in a sense, the GS [gross substitutes] condition is necessary to ensure existence of a Walrasian equilibrium". The "necessity" makes sense in their context in which any set of agents they consider may contain any agent who demands at most one unit of any good. (Since such agents have demand type $\mathcal{D}_{ss}^n$, their result is equivalent to the maximality of $\mathcal{D}_{ss}^n$ as a unimodular set (see Corollary 6.15).) But the specificity of the context in which claims like this makes sense often seems to be forgotten.

6.4.4 Equilibrium in extensions of the Product-Mix Auction

It is not hard to check that the bids in any Product-Mix Auction of the kind implemented by the Bank of England all represent strong substitutes preferences, so (see Section 6.4.2) equilibrium is guaranteed even if individual units of goods are indivisible. This remains true if the auction is augmented by permitting bidders to use "negative" bids (in which case, it can be shown that all strong substitutes preferences can be represented).[100]

In fact, the Bank's implementation of the Product-Mix Auction allows the auctioneer to ration whenever it wishes. However, there are many contexts in which rationing may not be possible. For example, a piece of radio spectrum may only be useful if it is above a certain minimum size. Similarly, bidders might make competing offers to build gas-fired plants, nuclear-power stations, wind farms, etc., to a government needing energy capacity—and nuclear-plants, at least, may be indivisible.[101] So results about equilibrium when goods are indivisible may be needed to apply the Product-Mix Auction to problems currently facing regulators such as the U.S. Federal Communications Commission, the U.K.'s Ofcom, and the U.K. Department for Energy and Climate Change. We can use Theorem 6.8 to guarantee that existence of competitive equilibrium is retained in extensions of the Product-Mix Auction—see Baldwin and Klemperer (in preparation).[102]

6.5 Existence of equilibrium for specific demands

Theorem 6.8 tells us which demand types *always* have a competitive equilibrium. When the answer is negative, it does not tell us whether competitive equilibrium exists for every supply bundle, for a specific set of demands. But if all intersections are "nice" (in the sense of Section 6.3) then we can apply Proposition A.4 to each intersection point to check for such a failure.

Take, for example, Agents 1 and 2 who have THs of the combinatorial types of Figures 1 and 12, respectively, and concave valuations. (A valuation function of the combinatorial type of Figure 1 *must* be concave. A valuation function of the type of Figure 12 need not be concave, though the specific valuation function of this type that is given in Example 2.10 is concave.)

The combinatorial type of aggregate demand will depend on how the agents' THs meet in price space; assume they only intersect "nicely". Applying Propositions 6.10 and A.4, we see that there exists a supply bundle such that competitive equilibrium does

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[100]See Klemperer (2010), and Baldwin and Klemperer (in preparation).

[101]Making matters even more complex, alternatives such as wind farms and back-up facilities for use when no wind is blowing may be complements.

[102]Product-Mix auctions can also be used to improve clock auctions: one criticism of clock auctions is that they fail to find the exact competitive equilibrium when it is unique, or the correct competitive equilibrium when it is not unique, see, for example, Harbord et al., 2011, Appendix A. (In auctions for substitutes the usual objective is to select the unique bidder-preferred competitive equilibrium in the event that competitive equilibrium is non-unique.) We can solve this problem of clock auctions by accepting Product-Mix bids either between the bid increments of a clock auction (this is a generalisation of the "intraround bidding" popularised by Ausubel and Cramton (2004)) and/or after using simpler techniques to identify the price range in which competitive equilibrium must lie. (Footnote 7 discusses other disadvantages of clock auctions and simultaneous multiple round auctions relative to (pure) Product-Mix auctions.)
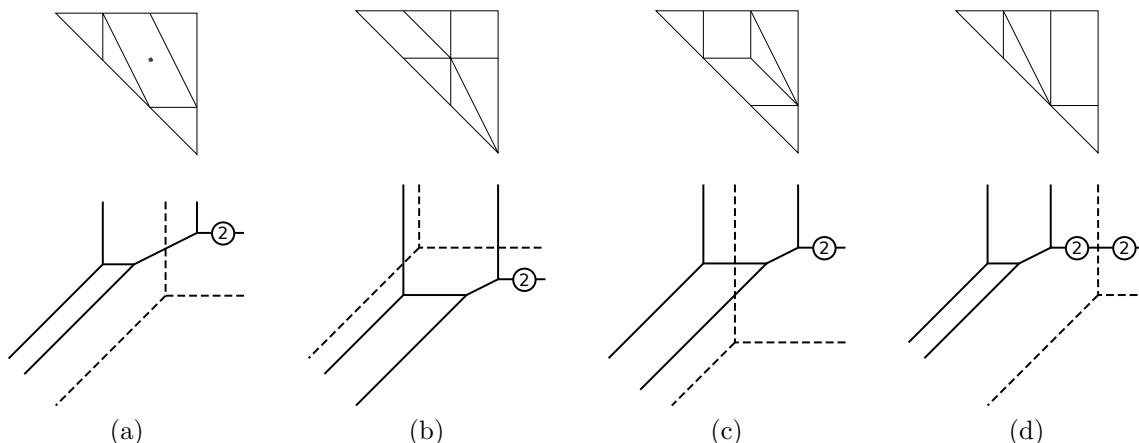
Figure 21: Examples of aggregate THs and SNPs of agents with THs of the combinatorial types of Fig. 1 (dashed line) and Fig. 12 (solid line). The number of intersections of the THs, weighted by facet weights, reveals the existence or failure of equilibrium.

not exist iff the facets with normals $(1, 0)$ and $(-1, 2)$ intersect (since $\det \begin{pmatrix} 1 & -1 \\ 0 & 2 \end{pmatrix} = 2 > 1$). An example of aggregate demand of this combinatorial type is illustrated in Figure 21a; the bundle $(1, 1)$ is in the interior of the parallelogram in the SNP of Figure 21a, and is never demanded on aggregate (see Proposition 6.10).

Combinatorial types of aggregate demand in which competitive equilibrium *does* exist for any supply bundle are illustrated in Figures 21b, 21c and 21d (there are others).

In Figures 21b and 21c, there are *two* intersections between the THs. In each case, the areas of the SNP faces corresponding to the intersections are 1. We call this area the 'multiplicity' of the intersection; note that it is, of course, the determinant of the (primitive integer) edges of the SNP face (and so, as we have seen, intimately connected with the existence of competitive equilibrium).

Conversely, in Figures 21a and 21d there is only *one* intersection. Now, however, the corresponding SNP face has area 2; we say the 'multiplicity' of the intersection is 2.

Observe that in each case, the number of intersections, weighted by multiplicity, is 2. It can be checked that this holds for every other aggregate of the demands of two agents whose individual THs are of the same combinatorial types as Figures 1 and 12, respectively. This is a special case of the *Tropical Bézout Theorem.*[103]

However, the natures of the multiplicity 2 intersections in Figures 21a and 21d are different. In Figure 21d, one of the corresponding facets is of weight 2; Agent 2 has a concave valuation and so has *three* bundles in its demand set, so Proposition 6.10 does not apply–the bundles 'inside' the weight-2 facet (in the centres of the long edges of the rectangle in the SNP) are both demanded at this price. The best way to understand this situation is that 'two intersections have become arbitrarily close'. By contrast, in Figure 21a, neither of the corresponding facets has weight 2, Proposition 6.10 *does* apply, and the bundle in the centre of the parallelogram is *not* demanded at any price.

Recall that the multiplicity of the intersection is the area of the SNP face, which

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[103]See Richter-Gebert et al (2005).

equals the (absolute value of the) determinant of its edges. The key point is that this can be factorised into the product of the facet weights times the (absolute value of the) determinant of the *primitive integer* edge directions (that is, the primitive integer facet normals). And equilibrium fails iff the (absolute value of the) *latter* determinant exceeds 1. So the *existence* of a supply bundle for which competitive equilibrium fails is signalled by a case in which the sum of intersections, weighted *only* by facet weights, is too small.

These ideas can be applied more generally, as will be developed in future work.

7   Conclusion

Studying the tropical geometry of demand yields a range of insights. The structure of an agent's preferences can be efficiently summarised by a set of vectors that is orthogonal to the divisions between the regions of price space in which the agent demands different bundles. So examining these vectors is an efficient way of determining the "type" of demand, and the same set of vectors also generates the surface of the convex hull of the agent's valuation function in quantity space. The duality between these representations has powerful implications, and the pictorial representations that tropical geometry gives us generate new intuitions.[104]

We began this work while studying the properties of many-dimensional Product-Mix Auctions. Convex and tropical geometry is the key to much of our analysis in Baldwin and Klemperer (in preparation) in which we describe ways in which different preferences can be represented in these auctions, and the implications of different restrictions on bids.[105]   Geometric reasoning has also helped us develop extensions to the Bank of England's original implementation of the auction,[106] and understand the connections to related auction designs.[107]

In other work, we have found that similar geometric analysis is useful in understanding results obtained by others, and that it can prove these results more quickly than currently-used techniques. So we are optimistic that tropical-geometric analysis will yield more economic insights in the future; we hope others will take up these methods.

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[104]These intuitions are obscured by existing pictorial representations which shoehorn indivisible demand into the standard divisible-demand framework.

[105]In the Bank of England's implementation, the bid-taker expresses preferences through a "supply function" while bidders can make sets of "or" bids that can, if desired, be represented as sets of points on a graph. Permitting negative as well as positive bids broadens the set of preferences that can be expressed, as does permitting bidders to specify additional constraints (Klemperer, 2008, 2010). The issue is: what kinds of bids should we permit to achieve a sufficiently rich representation of preferences, while retaining a unique solution (the extent to which we can permit some degree of complements is a particular challenge), achieving an efficient outcome (in particular, not incentivising strategic behaviour), and retaining simplicity and transparency?

[106]Extensions include broadening the range of contexts to which these (or related) auctions can be applied, through a better understanding of when equilibrium is guaranteed to exist, as well as better ways of representing bidders' and bid-takers' multi-dimensional preferences.

[107]Related designs include, in particular, the Assignment Auction suggested independently by Milgrom (2009), and versions of Simultaneous Multiple Round Auction (see, e.g., Milgrom, 2000) and "Clock Auctions" (see, e.g., Ausubel and Milgrom, 2002, Gul and Stacchetti, 2000, and Milgrom and Strulovici, 2009); see also the papers in Cramton, Shoham, and Steinberg (2006). As noted in the Introduction, we are also concerned with efficient solution techniques for Product-Mix Auctions, both when we need integer solutions, and when rationing is permitted, etc.

A   Proofs of Results in the text

A.1   Proofs of Results in Section 4

Proof of Theorem 4.4. First we show 1⇒2: Suppose u is of type $\mathcal{D}$. Generically the line $[\mathbf{p}, \mathbf{p}']$ crosses only facets, not any lower dimensional cells in $\mathcal{T}_u$. Furthermore, because the UDRs are open sets and because there are only finitely many cells of lower dimension than $n-1$, we can always choose a change in price, $\mathbf{q}$, such that $D_u(\mathbf{p} - \mathbf{q}) = D_u(\mathbf{p})$ and $D_u(\mathbf{p}' - \mathbf{q}) = D_u(\mathbf{p}')$ and the line $[\mathbf{p} - \mathbf{q}, \mathbf{p}' - \mathbf{q}]$ does indeed cross only facets; we choose $\mathbf{q}$ sufficiently small that $[\mathbf{p} - \mathbf{q}, \mathbf{p}' - \mathbf{q}]$ only crosses facets that are also crossed by $[\mathbf{p}, \mathbf{p}']$, although the latter may meet these facets at their boundaries. Now let $\mathbf{x}^0, \ldots, \mathbf{x}^l$ be demanded in each UDR that $[\mathbf{p} - \mathbf{q}, \mathbf{p}' - \mathbf{q}]$ meets; by construction, each of these bundles is also demanded at some price in sequence on the line $[\mathbf{p}, \mathbf{p}']$ and, also by construction, the difference between each pair of consecutive bundles is in the direction of a facet normal, that is, a vector in $\mathcal{D}$. Since the bundle demanded changes in each case, $(\mathbf{x}^j - \mathbf{x}^{j-1}).(\mathbf{p}' - \mathbf{p}) < 0$ in each case.

Next show 2⇒4: Suppose that $\{\mathbf{x}\} = D_u(\mathbf{p})$ and that $\mathbf{x} \notin D_u(\mathbf{p}')$. By assumption we can break down the demand change from $\mathbf{p}$ to $\mathbf{p}'$ in improving $\mathbb{Z}\mathcal{D}$-steps. Let $\mathbf{x}''$ be the first bundle $\mathbf{x}^1$ in this sequence; we know that $\frac{1}{w}(\mathbf{x}'' - \mathbf{x}) \in \mathcal{D}$ for some $w \in \mathbb{Z}$ and that $(\mathbf{p}' - \mathbf{p}).(\mathbf{x}'' - \mathbf{x}) < 0$. We re-write the latter as

$$(\mathbf{p}' - \mathbf{p}).\mathbf{x}'' < (\mathbf{p}' - \mathbf{p}).\mathbf{x} \tag{5}$$

Moreover, since $\mathbf{x}'' \in D_u((1 - \lambda_1)\mathbf{p} + \lambda_1 \mathbf{p}')$, we know

$$u(\mathbf{x}'') - [(1 - \lambda_1)\mathbf{p} + \lambda_1 \mathbf{p}'].\mathbf{x}'' \geq u(\mathbf{x}) - [(1 - \lambda_1)\mathbf{p} + \lambda_1 \mathbf{p}'].\mathbf{x}$$

Subtracting $(1 - \lambda_1)$ times equation (5) we obtain

$$u(\mathbf{x}'') - \mathbf{p}'.\mathbf{x}'' > u(\mathbf{x}) - \mathbf{p}'.\mathbf{x}$$

i.e. $\mathbf{x}''$ is strictly preferred to $\mathbf{x}$ at price $\mathbf{p}'$, as required.

Next we show 4⇒1. Suppose u is not of demand type $\mathcal{D}$. Then $\mathcal{T}_u$ has a facet F with primitive integer normal $\mathbf{n} \notin \mathcal{D}$. Let $\mathbf{p}^0$ be in the interior F° of this facet. For $\epsilon > 0$ sufficiently small, $\#D_u(\mathbf{p}^0 + \epsilon \mathbf{n}) = 1$; let $\{\mathbf{x}\} = D_u(\mathbf{p}^0 + \epsilon \mathbf{n})$. Then $\mathbf{x} \notin Du(\mathbf{p}^0 - \eta \mathbf{n})$ for any $\eta > 0$; set $\mathbf{p}' := \mathbf{p}^0 - \eta \mathbf{n}$ where η is sufficiently small that, at $\mathbf{p}'$, any bundle in $D_u(\mathbf{p}^0)$ is preferred to to any outside this set. The only bundles strictly preferred to $\mathbf{x}$ at price $\mathbf{p}'$ are bundles in $D_u(\mathbf{p}^0) - \{\mathbf{x}\}$. But, since $\mathbf{p}^0$ is in the interior of the facet F, we know that $D_u(\mathbf{p}^0) = \{\mathbf{x} + w\mathbf{n} \mid w \in W\}$ for some finite set $W \subsetneq \mathbb{Z}$. So since we assumed that $\mathbf{n} \notin \mathcal{D}$, the ordinary $\mathbb{Z}\mathcal{D}$-improvement property cannot hold.

It is clear that 2⇒3, so we conclude by showing that 3⇒1. So suppose that u satisfies 3. Consider a facet normal $\mathbf{v}$ of $\mathcal{T}_u$. Since $\mathbf{v} \neq 0$ we can pick i such that $v_i \neq 0$. Then it is possible to pick some $\epsilon > 0$ and prices $\mathbf{p}, \mathbf{p} + \epsilon \mathbf{e}^i$ in the UDRs either side of this facet, so that $[\mathbf{p}, \mathbf{p} + \epsilon \mathbf{e}^i]$ crosses only this facet. By property 3 we can break down the change in demand from $\mathbf{p}$ to $\mathbf{p} + \epsilon \mathbf{e}^i$ in improving $\mathbb{Z}\mathcal{D}$-steps – but it is clear that the only way to break down the change in demand from $\mathbf{p}$ to $\mathbf{p} + \epsilon \mathbf{e}^i$ is in steps in the direction of $\mathbf{v}$. We conclude that $\mathbf{v} \in \mathcal{D}$. Hence, u is of demand type $\mathcal{D}$. $\qquad\square$

Proof of Theorem 4.5. This proof closely follows the proof of Theorem 4.4, except that in showing 1⇒2 we choose $\mathbf{x}^0, \ldots, \mathbf{x}^l$ which are demanded on $[\mathbf{p} - \mathbf{q}, \mathbf{p}' - \mathbf{q}]$ but are not necessarily demanded in any UDR; instead, we stipulate that each new bundle differs from the former by a vector in $\mathcal{D}$. Since u is concave, such bundles exist. In showing 2⇒4 we note that, if we start with a series of improving $\mathcal{D}$-steps, then $\mathbf{x}''$ will differ from $\mathbf{x}$ by a vector in $\mathcal{D}$. That 4⇒1 and 3⇒1 clearly follow from 4⇒1 and 3⇒1 of Theorem 4.4, respectively, under the additional assumption that u is concave. □

Example A.1. Consider the unimodular demand type $\mathcal{D} = \{\pm(1,0), \pm(0,1), \pm(1,1)\}$ and the non-concave valuation of this type:

$x_1 = 3$	$x_1 = 2$	$x_1 = 1$	$x_1 = 0$	u
5	4	2	0	$x_2 = 0$
6	5	4	2	$x_2 = 1$
6	6	4	4	$x_2 = 2$
6	6	6	5	$x_2 = 3.$

We show that we *can* break down the change in demand between any $\mathbf{p}, \mathbf{p}'$ into improving $\mathcal{D}$-steps. This valuation fails to be concave at the bundle $(1,2)$; it would have the same TH but would be concave if $v(1,2)$ were equal to 5, in which case we would have $(1,2) \in D_u(1,1)$. By Theorem 5.11 and Corollary 5.5, we know that we would be able to break down the demand change from any $\mathbf{p}$ to any $\mathbf{p}'$ in improving $\mathcal{D}$-steps if it were concave. Thus there is only a question as to whether this is possible for $\mathbf{p}, \mathbf{p}'$ such that $(1,1) \in [\mathbf{p}, \mathbf{p}']$. And we need only consider $\mathbf{p}, \mathbf{p}'$ in UDRs and facets that contain $(1,1)$ in their closure.

It is easiest to break down the cases to consider by considering possible choices of $\mathbf{x}^0$. Suppose first that $(0,2) \in D_u(\mathbf{p})$. Then $(3,1) \in D_u(\mathbf{p}')$ and $p_1 - p'_1 > 0$ and $p_1 - p'_1 \geq p_2 - p'_2 \geq 0$. So we may take $\mathbf{x}^1 = (1,1)$, $\mathbf{x}^2 = (2,1)$, $\mathbf{x}^3 = (3,1)$ as the demand change broken down in improving $\mathcal{D}$-steps. And if $\mathbf{x}^0 = (3,1)$ then it is easy to see that the same steps in reverse will break down the demand change in improving $\mathcal{D}$-steps. Moreover, by the symmetry of the figure, we may find an analogous way to break down the demand change if $\mathbf{x}^0 = (2,0)$ or $(1,3)$.

Suppose next that $(1,1) \in D_u(\mathbf{p})$. Then $(2,2) \in D_u(\mathbf{p}')$ and $p_1 - p'_1 = p_2 - p'_2 > 0$. So if $(1,1) = \mathbf{x}^0$ then we may take $\mathbf{x}^1 = (2,1)$ and $\mathbf{x}^2 = (2,2)$ as the demand change broken down in improving $\mathcal{D}$-steps. Again, if $\mathbf{x}^1 = (2,2)$ then the reverse sequence will suffice.

Suppose that $(0,3) \in D_u(\mathbf{p})$. Then $(3,0) \in D_u(\mathbf{p}')$ and $p'_1 - p_1 \leq 0$, $p_2 - p'_2 \leq 0$ and both cannot hold with equality. Suppose the latter holds with equality then $(0,2) \in D_u(\mathbf{p})$ also. As argued above it follows that $(3,1) \in D_u(\mathbf{p}')$. In this case we break down the demand change in improving $\mathcal{D}$ steps via $(1,3), (2,2), (3,1)$. Similarly, if $p'_1 - p_1 = 0$ we can break down the demand change in improving $\mathcal{D}$-steps. Finally, if $p'_1 - p_1 < 0$ and $p'_2 - p_2 < 0$ then $(3,0) \in D_u(\mathbf{p}')$ and we can break down the demand change as $(0,3), (1,3), (2,2), (3,1), (3,0)$, these being improving $\mathcal{D}$-steps.

Example A.2 (Milgrom and Strulovici, 2009, p224.)**.** The valuation

$x_1 = 2$	$x_1 = 1$	$x_1 = 0$	u
2	1	0	$x_2 = 0$
3	1	1	$x_2 = 1$
4	3	2	$x_2 = 2$

is of demand type $\mathcal{D} = \{\pm(1,0), \pm(0,1), \pm(1,-1)\}$ and does satisfy the $\mathcal{D}$-improvement property, but is not concave.

Proof of Proposition 4.14 1. By definition, $\mathbf{x} \in D_u(\mathbf{p})$ if $\mathbf{p}^T(\mathbf{x} - \mathbf{x}') \leq u(\mathbf{x}) - u(\mathbf{x}')$ for all $\mathbf{x}' \in A$, with equality iff $\mathbf{x}' \in D_u(\mathbf{p})$ also. For any invertible matrix G, we may re-write

$$\mathbf{p}^T(\mathbf{x} - \mathbf{x}') = \mathbf{p}^T G G^{-1}(\mathbf{x} - \mathbf{x}') = (G^T\mathbf{p})^T(G^{-1}\mathbf{x} - G^{-1}\mathbf{x}').$$

If G is additionally unimodular, then $G^{-1}\mathbf{x}$ and $G^{-1}\mathbf{x}' \in \mathbb{Z}^n$. We define a new valuation G^*u on the finite set $G^{-1}A \subsetneq \mathbb{Z}^n$ via $G^*u(\mathbf{y}) := u(G\mathbf{y})$. If we write $\mathbf{y} = G^{-1}\mathbf{x}$ and $\mathbf{y}' = G^{-1}\mathbf{x}'$ then $(G^T\mathbf{p})^T(\mathbf{y} - \mathbf{y}') \leq G^*u(\mathbf{y}) - G^*u(\mathbf{y}')$ holds iff $\mathbf{p}^T(\mathbf{x} - \mathbf{x}') \leq u(\mathbf{x}) - u(\mathbf{x}')$. So we have

$$\mathbf{x} \in D_u(\mathbf{p}) \iff \mathbf{y} = G^{-1}\mathbf{x} \in D_{G^*u}(G^T\mathbf{p}),$$

as required.

2. Since the underlying set of $\mathcal{T}_u$ is those $\mathbf{p}$ for which $\#D_u(\mathbf{p}) > 1$ it follows immediately from 1. that $\mathcal{T}_{G^*u} = \{G^T\mathbf{p} \mid \mathbf{p} \in \mathcal{T}_u\}$, as required.

3. Suppose $\mathbf{v}$ is normal to a facet F of $\mathcal{T}_u$. It follows from 2. that the facet corresponding to F in $\mathcal{T}_{G^*u}$ has the form $G^T F = \{G^T\mathbf{p} \mid \mathbf{p} \in F\}$. We know $\mathbf{p}^T\mathbf{v}$ is constant for $\mathbf{p} \in F$, from which it follows that $(G^T\mathbf{p})^T G^{-1}\mathbf{v} = \mathbf{p}^T G G^{-1}\mathbf{v}$ is constant for $G^T\mathbf{p} \in G^T F$: we see $G^{-1}\mathbf{v}$ is normal to a facet of $\mathcal{T}_{G^*u}$. As G has an integer inverse, the converse is also true. Trivially, for any unimodular matrix G, the valuation G^*u is concave iff the valuation u is. $\qquad \square$

A.2 Proofs of Results in Section 5

Proof of Proposition 5.3. To show 1⇒2, take any such u, let $\mathbf{x} \in A$ and let $\mathbf{p}'$ be such that $\mathbf{x} \notin D_u(\mathbf{p}')$. Since u is concave there exists $\mathbf{p}$ such that $\mathbf{x} \in D_u(\mathbf{p})$. We can break down the demand change from $\mathbf{p}$ to $\mathbf{p}'$ in improving $\mathcal{D}$-steps; let $\mathbf{x}''$ be the first bundle $\mathbf{x}^1$ in this sequence. That $\mathbf{x}''$ has the desired properties follows exactly as in the proof that 2⇒4 in Theorem 4.4.

Now show that 2⇒1. Given any u of type $\mathcal{D}$, given $\mathbf{p} \in \mathbb{R}^n$, given any $\mathbf{x} \in D_u(\mathbf{p})$ and given any $\mathbf{p}' \neq \mathbf{p}$, either $\mathbf{x} \in D_u(\mathbf{p}')$, or there exists some maximal λ_1 such that $\mathbf{x} \in D_u((1 - \lambda_1)\mathbf{p} + \lambda_1\mathbf{p}')$. Suppose the latter is the case and choose $\lambda' > \lambda_1$ sufficiently small that all bundles in $D_u((1-\lambda_1)\mathbf{p}+\lambda_1\mathbf{p}')$ are preferred to any bundle outside this set, at price $\mathbf{p}'' := (1-\lambda')\mathbf{p}+\lambda'\mathbf{p}'$. If the complete $\mathcal{D}$-improvement property holds then there exists $\mathbf{x}^1$ such that $\mathbf{x}^1 - \mathbf{x} \in \mathcal{D}$ and such that $\mathbf{x}^1$ is strictly preferred to $\mathbf{x}$ at $\mathbf{p}''$. But, by our choice of $\mathbf{p}''$, we know $\mathbf{x}^1 \in D_u((1 - \lambda_1)\mathbf{p} + \lambda_1\mathbf{p}')$, and by the fact that $\mathbf{x}^1$ is strictly preferred to $\mathbf{x}$ at price $\mathbf{p}''$, we know the demand change from $\mathbf{x}$ to $\mathbf{x}^1$ satisfies the strict law of demand. So $\mathbf{x}^1$ is the first improving $\mathcal{D}$-step. If $\mathbf{x}^1 \notin D_u(\mathbf{p}'')$ we set $\lambda_2 = \lambda_1$ and find $\mathbf{x}^2$ strictly preferred to $\mathbf{x}^1$ at $\mathbf{p}''$ as before; by finiteness of the set A we eventually

find $\mathbf{x}^j \in D_u(\mathbf{p}'')$. Now let λ_{j+1} be maximal such that $\mathbf{x}^j \in D_u((1 - \lambda_{j+1})\mathbf{p} + \lambda_{j+1}\mathbf{p}')$ and continue as before; by finiteness of A we eventually retrieve $\mathbf{x}^l \in D_u(\mathbf{p}')$. $\qquad\square$

Proof of Proposition 5.6. 1. Given $\mathbf{p}'$ and $\mathbf{x} \in G^{-1}A$, $\mathbf{x} \notin D_{G^*u}(\mathbf{p}')$, we know (by Proposition 4.14.1) that $G\mathbf{x} \in D_u(G^{-T}\mathbf{p}')$. So there exists $\mathbf{y}''$ such that $u(\mathbf{y}''^{-T}\mathbf{p}'.\mathbf{y}'' > u(G\mathbf{x}) - G^{-T}\mathbf{p}'^{-1}\mathbf{x}$. and such that $\mathbf{y}'' - G\mathbf{x} \in \mathcal{D}$. Let $\mathbf{x}''^{-1}\mathbf{y}''$; then we have $u(G\mathbf{x}''^{-T}\mathbf{p}'^{-1}\mathbf{x}'' > u(G\mathbf{x}) - G^{-T}\mathbf{p}'^{-1}\mathbf{x}$, which says precisely that $G^*u(\mathbf{x}'') - \mathbf{p}'.\mathbf{x}''^*u(\mathbf{x}) - \mathbf{p}'.\mathbf{x}$. Moreover, $\mathbf{x}'' - \mathbf{x} = G^{-1}(\mathbf{y}'' - G\mathbf{x}) \in G^{-1}\mathcal{D}$. So G^*u satisfies the complete $G^{-1}\mathcal{D}$-improvement property. It is also concave (see Proposition 4.14.3), and so $G^{-1}\mathcal{D}$-complete, as required.

2 follows from 1 by definition. $\qquad\square$

Proof of Theorem 5.11. As noted in Footnote 62, we need only assume that for any vectors $\mathcal{V} \subseteq \mathcal{D}$, there exist linearly independent vectors $\mathbf{w}^1, \ldots, \mathbf{w}^s \in \mathcal{D}$, whose span over $\mathbb{R}$ coincides with the span over $\mathbb{R}$ of $\mathcal{V}$ and such that, for any $\mathbf{v}^1, \ldots, \mathbf{v}^{s-1} \in \mathcal{V}$ and any $i = 1, \ldots, s$, the vectors $\mathbf{v}^1, \ldots, \mathbf{v}^{s-1}, \mathbf{w}^i$ are either linearly dependent or are a unimodular set. This result is clearly implied by unimodularity of $\mathcal{D}$, as we may take $\mathbf{w}^1, \ldots, \mathbf{w}^k$ to be any maximal linearly independent subset of vectors in $\mathcal{D}$.

Suppose $\mathcal{D}$ is as above, and that u is concave and of demand type $\mathcal{D}$. We will show that u satisfies Condition 1 of Proposition 5.3. Let $\mathbf{p} \neq \mathbf{p}'$. Suppose $\mathbf{x} \in D_u(\mathbf{p})$. Let $\lambda_1 := \max\{\lambda \in [0,1] \mid \mathbf{x} \in D_u((1 - \lambda)\mathbf{p} + \lambda\mathbf{p}'\}$. If $\lambda_1 = 1$ then $l = 1$ and we are done; suppose not. Let $\mathbf{p}^1 := (1 - \lambda_1)\mathbf{p} + \lambda_1\mathbf{p}'$ and write $\mathbf{c}$ for $\mathbf{p}' - \mathbf{p}$.

We first argue that it is sufficient to find $\mathbf{v} \in \mathcal{D}$ such that $\mathbf{x}^1 := \mathbf{x} + \mathbf{v} \in D_u(\mathbf{p}^1)$ and such that $(\mathbf{p}' - \mathbf{p}).(\mathbf{x}^1 - \mathbf{x}) = \mathbf{c}.\mathbf{v} < 0$. Such $\mathbf{x}^1$ then satisfies the conditions to be the second improving $\mathcal{D}$-step. We may iterate the procedure; since A is finite, after a finite number of such steps find $\mathbf{x}^k$ such that $\mathbf{c}.\mathbf{x}^k$ is minimal for bundles in $D_u(\mathbf{p}^1)$. Then $\mathbf{x}^k \in D_u(\mathbf{p} + \epsilon\mathbf{c})$ for small enough $\epsilon > 0$. Setting now $\lambda_{k+1} := \max\{\lambda \in [0,1] \mid \mathbf{x}^k \in D_u((1 - \lambda)\mathbf{p} + \lambda\mathbf{p}')\}$ we continue as before; again by finiteness of A, this process must terminate at price $\mathbf{p}'$ after finitely many steps.

Let Δ be a minimal SNP face for u such that $\mathbf{x} \in \Delta$ and $\Delta \subseteq \text{Conv}\, D_u(\mathbf{p}^1)$ and such that $\mathbf{c}.\mathbf{y} < \mathbf{c}.\mathbf{x}$ for some $\mathbf{y} \in \Delta \cap \mathbb{Z}$. Such a face exists since $\text{Conv}\, D_u(\mathbf{p}^1)$ has the required properties with the possible exception of minimality. Let $\mathbf{w}^1, \ldots, \mathbf{w}^s \in \mathcal{D}$ be linearly independent vectors whose span over $\mathbb{R}$ coincides with the span over $\mathbb{R}$ of the edges of Δ and with the property as described at the beginning of this proof; since there exists $\mathbf{y} \in \Delta$ such that $\mathbf{c}.\mathbf{y} < \mathbf{c}.\mathbf{x}$ it follows that there exists i such that $\mathbf{c}.\mathbf{w}^i \neq 0$; we write $\mathbf{w}$ for whichever of $\mathbf{w}^i$ or $-\mathbf{w}^i$ provides $\mathbf{c}.\mathbf{w} < 0$ and show that $\mathbf{x} + \mathbf{w} \in D_u(\mathbf{p}^1)$.

Let $\bar{n} := \dim \Delta$ and let $\mathbb{A}^{\bar{n}}$ be the affine span of Δ. The polytope Δ is the intersection of half-spaces $H^+_{\mathbf{q},\alpha}$ and hyperplanes $H_{\mathbf{q},\alpha}$ where $(\mathbf{q}, \alpha) \in \mathbb{Z}$. We show that we may choose the normal vectors $\mathbf{q}$ so that $\mathbf{q}.\mathbf{w} = 0$ or ± 1 in every case.

If $\mathbf{q}.\mathbf{w} = 0$ then we have no problem, so assume not. For every $(\mathbf{q}, \alpha)$, there are $\bar{n} - 1$ edges of Δ embedded in $H_{\mathbf{q},\alpha}$ whose directions are $\bar{n} - 1$ linearly independent vectors in $\mathcal{D}$ (since u is of demand type $\mathcal{D}$). Let these directions be $\mathbf{v}^1, \ldots, \mathbf{v}^{\bar{n}-1} \in \mathcal{D}$. By assumption, $\mathbf{w}$ is not in the span of $\mathbf{v}^1, \ldots, \mathbf{v}^{\bar{n}-1}$. So $\mathbf{v}^1, \ldots, \mathbf{v}^{\bar{n}-1}, \mathbf{w}$ is a unimodular set (by definition of $\mathbf{w}$) and hence we may choose vectors $\mathbf{v}^{\bar{n}}, \ldots, \mathbf{v}^{n-1}$ (*not* necessarily in $\mathbf{D}$) such that $\det(\mathbf{v}^1, \ldots, \mathbf{v}^{n-1}, \mathbf{w}) = \pm 1$. But also, we may choose the halfspace $H^+_{\mathbf{q},\alpha}$ defining Δ so that $\mathbf{q}$ is additionally normal to all the vectors $\mathbf{v}^{\bar{n}}, \ldots, \mathbf{v}^{n-1}$: recall it was already normal to the first set of vectors identified, and the second set are by construction in directions linearly independent to Δ; we are free to choose the behaviour

of $H_{\mathbf{q},\alpha}$ beyond the affine span of Δ. However, if the vector $\mathbf{q}$ is a primitive integer vector normal to this set, then it is in the direction their vector product $\mathbf{v}^1 \times \cdots \times \mathbf{v}^{n-1}$; it is a theorem of linear algebra that $(\mathbf{v}^1 \times \cdots \times \mathbf{v}^{n-1}).\mathbf{w} = \det(\mathbf{v}^1, \ldots, \mathbf{v}^{n-1}, \mathbf{w})$. So, we can in each case choose $\mathbf{q}$ so that $\mathbf{q}.\mathbf{w} = \pm 1$.

We now wish to check that $\mathbf{x} + \mathbf{w} \in \Delta$. Since $\mathbf{x} \in \Delta$ we know that $\mathbf{q}.\mathbf{x} \geq \alpha$ and so, that $\mathbf{q}.(\mathbf{x} + \mathbf{w}) \geq \alpha + \mathbf{q}.\mathbf{w}$ for every defining halfspace $H_{\mathbf{q},\alpha}^+$ or hyperplane $H_{\mathbf{q},\alpha}$. For those $\mathbf{q}$ such that $\mathbf{q}.\mathbf{w} = 0$ we are done. Since $\mathbf{w}$ lies in the span over $\mathbb{R}$ of the edges of Δ, those $\mathbf{q}$ such that $\mathbf{q}.\mathbf{w} = \pm 1$ must define half-spaces and not hyperplanes. If $\mathbf{x} \in \Delta^\circ$ then in every such case $\mathbf{q}.\mathbf{x} > \alpha$ and hence, since the equations are integral, $\mathbf{q}.\mathbf{x} \geq \alpha + 1$; thus $\mathbf{q}.(\mathbf{x} + \mathbf{w}) \geq \alpha$. Our only difficulty arises if $\mathbf{x}$ lies in a (strict) face Δ' of Δ.

Suppose then that $\mathbf{x}$ is in such a face $\Delta' \subsetneq \Delta$. We show that $x \in \Delta'^\circ$ and that $\dim \Delta' = \dim \Delta - 1$ and $\Delta' \subset H_{\mathbf{c},\mathbf{c}.\mathbf{x}}$, where we recall that we write $\mathbf{c}$ for the change in price $\mathbf{p}' - \mathbf{p}$ under consideration.

First note that, by minimality of Δ, we know $\mathbf{c}.\mathbf{y} \geq \mathbf{c}.\mathbf{x}$ for all $\mathbf{y} \in \Delta'$. So $\Delta' \subset H_{\mathbf{c},\mathbf{c}.\mathbf{x}}^+$; let Δ'' be the face (which we will see to be the whole of Δ') of Δ' given by $\Delta' \cap H_{\mathbf{c},\mathbf{c}.\mathbf{x}}$.

Suppose for a contradiction that $\dim \Delta'' \leq \dim \Delta - 2$. By the standard properties of polytopes we know that Δ'' is also a face of Δ, and additionally is a face of the polytope $\overline{\Delta} := \Delta \cap H_{\mathbf{c},\mathbf{c}.\mathbf{x}}^-$. The latter polyhedron has the same dimension as Δ since, by assumption, Δ is not contained in $H_{\mathbf{c},\mathbf{c}.\mathbf{x}}$. So $\dim \Delta'' \leq \dim \overline{\Delta} - 2$, and so Δ'' is the intersection of at least two maximal strict faces of $\overline{\Delta}$. But at most one of these may be contained in the hyperplane $H_{\mathbf{c},\mathbf{c}.\mathbf{x}}$; the other must contain $\mathbf{y}$ such that $\mathbf{c}.\mathbf{y} \neq \mathbf{c}.\mathbf{c}$ and hence (by definition of $\overline{\Delta}$) must contain $\mathbf{y}$ such that $\mathbf{c}.\mathbf{y} < \mathbf{c}.\mathbf{x}$. But, since $\mathbf{x} \in \Delta''$ which is contained in this face, this contradicts the minimality assumption on Δ.

Thus $\dim \Delta'' \geq \dim \Delta - 1$; since $\Delta'' \subseteq \Delta' \subsetneq \Delta$ we conclude that $\Delta'' = \Delta'$, whence $\dim \Delta' = \dim \Delta - 1$ and $\Delta' \subseteq H_{\mathbf{c},\mathbf{c}.\mathbf{x}}$. Finally, if $\mathbf{x}$ were in the boundary of Δ' then $\mathbf{x}$ would be in a lower dimensional face of Δ than Δ', which we have shown not to be the case. So $\mathbf{x} \in \Delta'^\circ$.

We may now modify the list of halfspaces describing Δ: replace the half-space defining the maximal face Δ' with $H_{-\mathbf{c},-\mathbf{c}.\mathbf{x}}^+$. By assumption $\mathbf{c}.\mathbf{w} < 0$ and so $-\mathbf{c}.(\mathbf{x} + \mathbf{w}) \geq -\mathbf{c}.\mathbf{x}$. By construction, this was the remaining half-space defining Δ which required this check, and so we know that $\mathbf{x} + \mathbf{w} \in \Delta \subseteq \operatorname{Conv} D_u(\mathbf{p}^1)$. Moreover, as both vectors are integral, $\mathbf{x} + \mathbf{w} \in \operatorname{Conv} D_u(\mathbf{p}^1) \cap \mathbb{Z}^n$, and

since u is concave, it follows that $\mathbf{x} + \mathbf{w} \in D_u(\mathbf{p}^1)$, as required. $\qquad \square$

Example A.3. Consider the unimodular demand type $\mathcal{D} = \{\pm(1,0), \pm(0,1)\}$ and the non-concave valuation of this type:

$x_1 = 2$	$x_1 = 1$	$x_1 = 0$	u
2	1	0	$x_2 = 0$
4	2	2	$x_2 = 1$

We show that we *cannot* break down the demand change from every $\mathbf{p}$ to every other $\mathbf{p}'$ in improving $\mathbb{Z}\mathcal{D}$-steps. Consider the price change from $(1,3)$ to $(1,1)$ (noting that $(1,3)$ is not a UDR price; $D_u(1,3) = \{(0,0),(1,0),(2,0)\}$). The bundle $(1,0)$ is in $D_u(1,3)$, but $D_u(1,1) = \{(0,1),(2,1)\}$. We cannot break down the change with this starting bundle without changing x_1. However, the only vectors in $\mathbb{Z}\mathcal{D}$ which would give a change in x_1 are integer multiples of $\pm(1,0)$. The price change we consider is $(0,-2)$

so a change in direction $\pm(1,0)$ would not obey the strict law of demand. Thus, it is impossible to break down the change in demand from price $(1,3)$ to $(1,1)$ in improving $\mathbb{Z}\mathcal{D}$-steps.

Proof of Theorem 5.16. We follow the technique of Veblen and Franklin (1921) to show that $\mathcal{D}^n_{ss}$ is unimodular. Note first that any vector $\mathbf{v} = \mathbf{e}^i - \mathbf{e}^j$ satisfies $\mathbf{v}.\mathbf{1} = 0$, where $\mathbf{1} = (1, 1, \ldots, 1)^T$. So any set of n vectors that are all of the form $\mathbf{e}^i - \mathbf{e}^j$ does not have $\mathbf{1}$ in its span, so is not linearly independent and therefore has determinant 0. It follows that any set of linearly independent vectors in $\mathcal{D}^n_{ss}$ must include a coordinate vector $\mathbf{e}^i$ (or $-\mathbf{e}^i$). Now observe that the determinant of any matrix which has this set of vectors as its columns is non-zero (since the vectors are linearly independent), and also ± 1 times the $(n-1) \times (n-1)$ matrix formed when we delete row i and the column in which $\pm\mathbf{e}^i$ was placed. But since this $(n-1) \times (n-1)$ matrix therefore has non-zero determinant, its columns are linearly independent, and they are also vectors in $\mathcal{D}^{n-1}_{ss}$. So $\mathcal{D}^n_{ss}$ satisfies the determinant condition if $\mathcal{D}^{n-1}_{ss}$ does. But it is trivial that $\mathcal{D}^1_{ss}$ satisfies the condition so, by induction on n, $\mathcal{D}^n_{ss}$ is unimodular for all n.

We showed in the text that $\mathcal{D}^n_{ss}$ is maximal. $\qquad\square$

Proof of Lemma 5.19. 1. If u is of some demand type $\mathcal{D}$ as described, then we can break down the demand change from any $\mathbf{p}$ such that $\#D_u(\mathbf{p}) = 1$ to $\mathbf{p} + \epsilon\mathbf{e}^k$ in improving $\mathbb{Z}\mathcal{D}$-steps; that is, in each case, $\mathbf{x}^{j+1} - \mathbf{x}^j = w\mathbf{v}$ for $w \in \mathbb{Z}_+$ and $\mathbf{v} \in \mathcal{D}$, and additionally, at each such step, $(\mathbf{x}^{j+1} - \mathbf{x}^j).\mathbf{e}^k < 0$. It follows that $v_k < 0$ for this $\mathbf{v}$ and so, by assumption, that either $-\mathbf{v} \in \mathbb{Z}^n_{\geq 0}$ or $\sum_{i=1}^n v_i = 0$; we conclude that $\sum_{i=1}^n v_i \leq 0$. Thus $\sum_i x_i^{j+1} \leq \sum_i x_i^j$, as required. Applying this at each $\mathbb{Z}\mathcal{D}$-step provides the ordinary law of aggregate demand.

Conversely, if u is not of some demand type $\mathcal{D}$ as described, then u must have a facet with normal $\mathbf{v}$ not satisfying the description given. Both $\mathbf{v}$ and $-\mathbf{v}$ are facet normals so without loss of generality assume that $\sum_{i=1}^n v_i > 0$ but that there exists $k \in \{1, \ldots, n\}$ such that $v_k < 0$. It follows that $\mathbf{e}^k.\mathbf{v} \neq 0$ so there exist prices $\mathbf{p}, \mathbf{p} + \epsilon\mathbf{e}^k$ in the UDRs on either side of this facet; by construction the ordinary law of aggregate demand fails for these prices.

Finally, if $\mathcal{D}$ is unimodular and u is concave then we can break the demand change from *any* $\mathbf{p}$ to any $\mathbf{p} + \epsilon\mathbf{e}^k$ in improving $\mathcal{D}$-steps, and so the complete law of aggregate demand holds.

2. Suppose u is of concave demand type $\mathbf{D} \subset \{-1, 0, 1\}^n$. Then, for any $\mathbf{p}$ and any good i, we can move from those points in $D_u(\mathbf{p})$ minimal for i to those points maximal for i along the edge vectors; as u is concave every lattice point on the edges in also in $D_u(\mathbf{p})$, so we can move in steps entirely from $\mathbf{D}$ while staying in $D_u(\mathbf{p})$; since $\mathbf{D} \subset \{-1, 0, 1\}^n$ this illustrates the consecutive integer property.

Suppose that u is not of any demand type $\mathcal{D} \subset \{-1, 0, 1\}^n$; it follows that $\mathcal{T}_u$ has a facet with normal v such that $\|v_i\| \geq 2$ for some good i. For any $\mathbf{p}$ in the interior of this facet, the consecutive integer property fails for good i. $\qquad\square$

A.3 Proofs of Results in Section 6.1

Proof of Propositions 6.1 and 6.2. Proposition 6.1 is straightforward. Note that

$$\sum_{j \in J} \max_{\mathbf{x}^j \in A} \{u^j(\mathbf{x}^j) - \mathbf{p}.\mathbf{x}^j\} = \max \left\{ \sum_{j \in J} u^j(\mathbf{x}^j) - \mathbf{p}. \left(\sum_{j \in J} \mathbf{x}^j \right) \mid \mathbf{x}^j \in A^j, j \in J \right\},$$

and on the other hand (since $\mathbf{y} \in A$ iff $\mathbf{y} = \sum_{j \in J} \mathbf{x}^j, \mathbf{x}^j \in A_J$) that

$$\max_{\mathbf{y} \in A} \{U(\mathbf{y}) - \mathbf{p}.\mathbf{y}\}$$

$$= \max \left\{ \max \left\{ \sum_{j \in J} u^j(\mathbf{x}^j) \mid \mathbf{x}^j \in A_j, \sum_{j \in J} \mathbf{x}^j = \mathbf{y} \right\} - \mathbf{p}.\mathbf{y} \mid \mathbf{y} = \sum_{j \in J} \mathbf{x}^j, \mathbf{x}^j \in A_j j \in J \right\}$$

$$= \max \left\{ \sum_{j \in J} u^j(\mathbf{x}^j) - \mathbf{p}. \left(\sum_{j \in J} \mathbf{x}^j \right) \mid \mathbf{x}^j \in A, j \in J \right\},$$

and that the same arguments $\mathbf{x}^j \in A$, with $\mathbf{y} = \sum_{j \in J} \mathbf{x}^j$, are maximising in either case.

The text showed the underlying sets of $\mathcal{T}_U$ and $\mathcal{T}_{\{u^j\}}$ are the same, so completing the proof of Proposition 6.2 only requires checking the weightings are the same. So suppose F is a facet of $\mathcal{T}_U$ with adjacent UDRs U and U'; let $\mathbf{v}_F$ be a primitive integer vector pointing from U to U'. Suppose agent j demands $\mathbf{x}^j$ in U and $\mathbf{x}^{j'}$ in U' (for some agent these will be distinct, but not necessarily for all). Then $w_j(F)\mathbf{v}_F = \mathbf{x}^{j'} - \mathbf{x}^j$ for all j, and so

$$\sum_j w_j(F)\mathbf{v}_F = \sum_j \mathbf{x}^{j'} - \sum_j \mathbf{x}^j.$$

So $w_U(F) = \sum_j w_j(F) = w_{\{u^j\}}(F)$, as required. □

Proof of Proposition 6.5. Suppose $G^{-1}\mathcal{D}$ always has a competitive equilibrium. Consider any agent valuations $u^1, \dots, u^k$ of type $\mathcal{D}$ and let $\mathbf{x}$ be in the domain of their aggregate valuation. Then demands $G^*u^1, \dots, G^*u^k$ have type $G^{-1}\mathcal{D}$ and $\mathbf{y} := G^{-1}\mathbf{x}$ is in the domain of their aggregate valuation. By assumption competitive equilibrium exists in the latter case: there exists a price $\mathbf{p}$ at which the agent with valuation G^*u^i demands $\mathbf{y}^i$ and $\sum_i \mathbf{y}^i = \mathbf{y}$. But then in each case we may define $\mathbf{x}^i := G\mathbf{y}^i \in D_{u^i}(G^{-T}\mathbf{p})$ (see Proposition 4.14.1). At price $G^{-T}\mathbf{p}$ the market clears for $\mathbf{x} := \sum_i \mathbf{x}^i$. So $\mathcal{D}$ has a competitive equilibrium. The converse is shown by repeating the argument, using the unimodular matrix G^{-T}. □

A.4 Proof of results in Section 6.3

This Appendix gives the additional details needed to complete the proof of Theorem 6.8. Proposition A.4 shows that, for "nice" intersections, the condition of unimodularity is necessary and sufficient for competitive equilibrium to always exist.

The second half of the proof of the sufficiency part of the theorem shows that generically all TH intersections are "nice", and that any non-"nice" intersection is therefore close enough to being a "nice" intersection that Theorem 6.8's condition still suffices.

Lemmas A.5, A.8 and A.9 demonstrate that generically all single-point intersections of the TH are "nice". The logic is as follows: first (Lemma A.5), we show how to perform affine translations of agents' THs, and bound the associated change in valuation. Now consider an intersection of two cells from distinct agents' THs. Generically (in the space of affine translations) there can be no vector normal to both; if there were, a small shift of one of the agents' demands in the direction of this vector would mean the cells no longer intersected at all. We argue thus in Lemma A.8.

In Lemma A.9 we show how to make all intersections 'nice', while bounding the change in any agent's valuation. Begin by considering an intersection of two cells from distinct agents THs. If necessary, make small shifts as described by Lemma A.8. Now, for each of the two cells that intersect, we nominate a linearly independent set of vectors normal to adjacent facets. The fact that there is no vector normal to both the cells means that the union of these sets remains linearly independent. But the intersection of the two cells is now a cell of the TH of the aggregate demand of the two agents, and the collection of vectors we have defined so far are normal to facets in this TH whose intersection is this new cell. Continuing to add any additional agents' demands that intersect the cell generically, we can construct a set of linearly independent vectors, each normal to a facet of the TH of aggregate demand, such that the intersection of these facets locally defines the intersection of the cells in question.

After these small perturbations, any bundle is demanded at some price (by Proposition A.4). We complete the proof of Theorem 6.8 by showing that, if a bundle is demanded following an extremely small perturbation in agents' valuations, it must have also been demanded before this perturbation.[108] This proves the sufficiency of unimodularity (with concavity) for Theorem 6.8.

Proposition A.4. *Suppose price* $\mathbf{p}$ *is in the interior of an* $(n - k_i)$-*cell* C_i *of the TH* $\mathcal{T}_{u^i}$ *of each of* s *agents* $i = 1, \ldots, s$, *who have concave valuations* u^i, *and together have aggregate valuation* $\tilde{U}$. *Then every integer bundle in* $\mathrm{Conv}\, D_{\tilde{U}}(\mathbf{p})$ *is demanded at* $\mathbf{p}$ *if*

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[108]In more detail: consider an integer bundle that is "hidden" in the convex hull of aggregate demand at a price point in a *not*-nice intersection. If it is not demanded at this price, agents' aggregate utility from this bundle, at this price vector, must be strictly lower than their aggregate utility from any bundle that is demanded at this price. Since this bundle is a convex combination of other bundles that are demanded at this price vector, the aggregate valuation from the bundle in question is strictly lower than the same convex combination of the aggregate valuations of these other bundles. Let this aggregate valuation difference be ϵ.

Now consider perturbing all agents' valuation functions by arbitrarily small amounts, so that their TH undergoes a small translation in price space. It is straightforward, although somewhat tedious, to show that generically all the TH intersections are now "nice". So we can choose these small perturbations so this holds; additionally, we ensure that no agent's valuation of any available bundle is affected by more than $\frac{\epsilon}{3m}$, in which m is the number of agents present.

If $\mathcal{D}$ is concave and unimodular, the bundle in question is (by Proposition A.4) demanded by agents with the perturbed valuation functions at some price. But the perturbation of the valuation functions cannot change the aggregate valuation from either this bundle, or the same convex combination of the aggregate valuation of the other bundles, by more than $\epsilon/3$. So the aggregate valuation from this bundle is still below the same convex combination of the aggregate valuation of the other bundles, and therefore the aggregate utility of this bundle is also still below the same convex combination of the aggregate utility of the other bundles at any prices (since at any prices, the cost of this bundle equals this convex combination of the cost of the other bundles). So we have a contradiction, and the lattice point must have been demanded at the original price point. That is, Theorem 6.8's condition is also sufficient for non-"nice" intersections.

each C_i is a subset of the intersection of a set of facets $F_1^i, \dots, F_{k_i}^i$ of $\mathcal{T}_{u^i}$ (not necessarily comprising all facets of $\mathcal{T}_{u^i}$ that pass through C_i) with primitive integer normal vectors $\mathbf{v}_1^i, \dots, \mathbf{v}_{k_i}^i$ and $\{\mathbf{v}_j^i \mid i = 1, \dots, s; j = 1, \dots, k_i\}$ are unimodular.

Proof. All bundles demanded by agent i at $\mathbf{p}$ are demanded throughout the $(n - k_i)$-cell C_i, which corresponds to a k_i-dimensional polytope Δ_i in the SNP of agent i. Moreover, Δ_i possesses an edge in direction $\mathbf{v}_j^i$ for $j = 1, \dots, k_i$; each corresponds to the facet F_j^i. Thus, if $\mathbf{y}^i$ is some integer bundle in $D_{u^i}(\mathbf{p})$, then (by a dimension count) the affine span of Δ_i is precisely $\left\{ \mathbf{y}^i + \sum_{j=1}^{k_i} \beta_j^i \mathbf{v}_j^i \mid \beta_j^i \in \mathbb{R} \text{ for } j = 1, \dots, k_i \right\}$, and in particular, $D_{u^i}(\mathbf{p})$ is contained in this set.

Thus, using equation (4) we may express aggregate demand among these agents as $D_{\tilde{U}}(\mathbf{p}) = \left\{ \mathbf{y} + \sum_{i=1}^{s} \sum_{j=1}^{k_i} a_j^i \mathbf{v}_j^i \mid \mathbf{y}^i + \sum_{j=1}^{k_i} a_j^i \mathbf{v}_j^i \in D_{u^i}(\mathbf{p}) \text{ for } i = 1, \dots, s \right\}$, where $\mathbf{y} := \sum_{i=1}^{s} \mathbf{y}^i$.

Now, suppose $\mathbf{x}$ is an integer bundle in $\operatorname{Conv} D_{\tilde{U}}(\mathbf{p})$. Then $\mathbf{x} - \mathbf{y}$ is in the span of the $\mathbf{v}_j^i$. But since they are an integer basis for their span, we can write $\mathbf{x} - \mathbf{y} = \sum_{i=1}^{s} \sum_{j=1}^{k_i} b_j^i \mathbf{v}_j^i$, for some $b_j^i \in \mathbb{Z}$. So we can define $\mathbf{x}^i := \mathbf{y}^i + \sum_{j=1}^{k_i} b_j^i \mathbf{v}_j^i$, and know that $\mathbf{x}^i \in \mathbb{Z}^n$.

But we also know $\mathbf{x}^i \in \operatorname{Conv} D_{u^i}(\mathbf{p})$. To see this, observe that since $\mathbf{x} \in \operatorname{Conv} D_{\tilde{U}}(\mathbf{p})$, we can write $\mathbf{x} - \mathbf{y} = \sum_{\beta} \sum_{i=1}^{s} \sum_{j=1}^{k_i} \lambda_\beta a_{j,\beta}^i \mathbf{v}_j^i$ for some finite set of weights $\lambda_\beta \in [0, 1]$ such that $\sum_{\beta} \lambda_\beta = 1$ and such that $\mathbf{y}^i + \sum_{j=1}^{k_i} a_{j,\beta}^i \mathbf{v}_j^i \in D_{u^i}(\mathbf{p})$ for each agent i and for each β. But since the $\mathbf{v}_j^i$ are linearly independent, there is an unique way to write $\mathbf{x} - \mathbf{y}$ as a weighted sum of the $\mathbf{v}_j^i$, so $b_j^i = \sum_{\beta} \lambda_\beta a_{j,\beta}^i$, and so $\mathbf{x}^i = \mathbf{y}^i + \sum_{j=1}^{k_i} b_j^i \mathbf{v}_j^i = \mathbf{y}^i + \sum_{j=1}^{k_i} \sum_{\beta} \lambda_\beta a_{j,\beta}^i \mathbf{v}_j^i \in \operatorname{Conv} D_{u^i}(\mathbf{p})$.

So $\mathbf{x}^i$ is an integer vector in $\operatorname{Conv} D_{u^i}(\mathbf{p})$. By concavity of u^i there exists *some* price at which $\mathbf{x}^i$ is demanded by agent i (Lemma 2.5), and so by Lemma 2.6 we know $\mathbf{x}^i \in D_{u^i}(\mathbf{p})$. Thus $\mathbf{x} = \sum_{i=1}^{s} \mathbf{x}^i \in D_{\tilde{U}}(\mathbf{p})$. That is, $\mathbf{x}$ is demanded at $\mathbf{p}$, as required. $\qquad \square$

We introduce the affine perturbations discussed above.

Lemma A.5. *Suppose an agent has valuation function $u : A \to \mathbb{R}$. For any $\mathbf{w} \in \mathbb{R}^n$, we may define a valuation function $u_{\mathbf{w}} : A \to \mathbb{R}$ such that, for all $\mathbf{p} \in \mathbb{R}^n$, we have*

1. $D_{u_{\mathbf{w}}}(\mathbf{p}) = D_u(\mathbf{p} + \mathbf{w})$;

2. $\mathcal{T}_{u_{\mathbf{w}}} = \{\mathbf{p} - \mathbf{w} \mid \mathbf{p} \in \mathcal{T}_u\}$;

3. $\|u_{\mathbf{w}}(\mathbf{x}) - u(\mathbf{x})\| \le R\|\mathbf{w}\|$, where R satisfies $\|\mathbf{x}\| < R$ for all $\mathbf{x} \in A$.

Proof. Let $u_{\mathbf{w}}(\mathbf{x}) = u(\mathbf{x}) - \mathbf{x}.\mathbf{w}$. Then

$$D_{u_{\mathbf{w}}}(\mathbf{p}) = \arg\max_{\mathbf{x} \in A} \{u(\mathbf{x}) - \mathbf{x}.\mathbf{w} - \mathbf{x}.\mathbf{p}\} = \arg\max_{\mathbf{x} \in A} \{u(\mathbf{x}) - \mathbf{x}.(\mathbf{p} + \mathbf{w})\} = D_u(\mathbf{p} + \mathbf{w}).$$

The remainder of the lemma follows by definition of $\mathcal{T}_u$, and the Cauchy-Schwarz inequality. $\qquad \square$

To prove that the hypotheses of Proposition A.4 are satisfied after such perturbations, is convenient to use "annihilator spaces". For a linear or affine subspace of $\mathbb{R}^n$, these

give the linear subspace of all orthogonal vectors. We recall their definition and basic properties.

Definition A.6 (See e.g. Spence et al., 2000)**.** If $C \subseteq \mathbb{R}^n$ is an affine subspace, define

$$C^\circ := \{\mathbf{v} \in \mathbb{R}^n \mid \mathbf{v}.(\mathbf{c} - \mathbf{c}') = 0, \ \forall \mathbf{c}, \mathbf{c}' \in C\}.^{109}$$

Note that if $D = C + \mathbf{w}$ for some $\mathbf{w} \in \mathbb{R}^n$ then $D^\circ = C^\circ$.

We use annihilator spaces for the following results.

Lemma A.7 (See e.g. Spence et al., 2000)**.** *Suppose that $C_1, C_2 \subseteq \mathbb{R}^n$ are affine subspaces.*

1. If $C_1 \subseteq C_2$ then $C_2^\circ \subseteq C_1^\circ$

2. If $C_1 \cap C_2 \neq \emptyset$ then additionally $(C_1 \cap C_2)^\circ = C_1^\circ + C_2^\circ$.

3. $\dim C_1 + \dim(C_1)^\circ = n$

Proof. Part 1 is clear. Part 2 follows from the standard result when C_1 and C_2 are linear subspaces (see, e.g. Spence et al. 2000): if $-\mathbf{w} \in C_1 \cap C_2$ then $C_1 + \mathbf{w}$ and $C_2 + \mathbf{w}$ are linear subspaces, and so $((C_1 + \mathbf{w}) \cap (C_2 + \mathbf{w}))^\circ = (C_1 + \mathbf{w})^\circ + (C_2 + \mathbf{w})^\circ$. But $(C_1 + \mathbf{w}) \cap (C_2 + \mathbf{w}) = (C_1 \cap C_2) + \mathbf{w}$ so the result follows from the note above. Part 3 similarly follows immediately from the linear case. $\square$

Now we show that any two THs may be perturbed so that the intersection of their cells is 'generic' (as given in the statement of the following lemma):

Lemma A.8. *Suppose we have agents 1 and 2 with valuation functions u^1 and u^2 (not necessarily concave). For any $\epsilon > 0$ we may find a vector $\mathbf{w}$ such that, if we perturb agent 2's demand by $\mathbf{w}$ to obtain $u_\mathbf{w}^2$, then $\|u_\mathbf{w}^2(\mathbf{x}) - u^2(\mathbf{x})\| < \epsilon$ for all $\mathbf{x} \in A$, and any cells C_1 of $\mathcal{T}_{u^1}$ and $C_2^\mathbf{w}$ of $\mathcal{T}_{u_\mathbf{w}^2}$ satisfy $C_1 \cap C_2^\mathbf{w} \neq \emptyset \Rightarrow C_1^\circ \cap (C_2^\mathbf{w})^\circ = \{0\}$.*

Proof. Suppose that C_1 in $\mathcal{T}_{u^1}$ and C_2 in $\mathcal{T}_{u^2}$ satisfy $C_1 \cap C_2 \neq \emptyset$ and $C_1^\circ \cap C_2^\circ \neq \{0\}$. Choose $\mathbf{w}_1 \in C_1^\circ \cap C_2^\circ$ with $\mathbf{w}_1 \neq 0$. Then, for all $\eta > 0$, we show that $(C_2 + \eta \mathbf{w}_1) \cap C_1 = \emptyset$. For, given any $c_2 \in C_2$, if $c_1 \in C_1 \cap C_2$ then $\mathbf{w}_1.(c_1 - (c_2 + \eta \mathbf{w}_1)) = \eta \|\mathbf{w}_1\|^2 \neq 0$ (since $c_1, c_2 \in C_2$) and so, since $\mathbf{w}_1 \in C_1^\circ$, it follows that $c_2 + \eta \mathbf{w}_1 \notin C_1$.

On the other hand, recall that the cells of THs are closed objects. It follows that a sufficiently small perturbation of one of the THs will not introduce any new intersections between cells. So there exists $\eta_1 > 0$ such that if $\eta < \eta_1$ then no new intersections arise.[110]

Since THs consist of a finite number of affine cells, we may suppose that there are in total d intersections of cells in $\mathcal{T}_{u^1}$ and $\mathcal{T}_{u^2}$ whose annihilator spaces have non-zero intersection. We find $\mathbf{w}_j$ and η_j as above for each in turn, and apply them all.[111] Thus,

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[109]If $D \leq \mathbb{R}^n$ is a linear subspace then this definition clearly coincides with the usual $D^\circ := \{\mathbf{v} \in \mathbb{R}^n \mid \mathbf{v}.\mathbf{d} = 0 \ \forall \mathbf{d} \in D\}$.

[110]To be precise: if $C_{2a}^\mathbf{w}$ in $\mathcal{T}_{u_\mathbf{w}^2}$ satisfies $C_{2a}^\mathbf{w} \cap C_{1a} \neq \emptyset$ for any cell C_{1a} in $\mathcal{T}_{u^1}$ then the corresponding C_{2a} in $\mathcal{T}_{u^2}$ satisfies $C_{2a} \cap C_{1a} \neq \emptyset$.

[111]Strictly speaking, each η_j should be found when we compare the cells after $\mathcal{T}_{u^2}$ has undergone the translations corresponding to intersections $1, \dots, j - 1$.

perturbing Agent 2 by $\mathbf{w} = \eta\mathbf{v}$, where $\mathbf{v} = \sum_{j=1}^d \eta_j\|\mathbf{w}_j\|$ and $\eta \in (0,1]$, gives us the intersection properties required. To ensure that the perturbation to the agent's valuation is sufficiently small, we choose $\eta < \frac{\epsilon}{R\|\mathbf{v}\|}$ where R satisfies $\|\mathbf{x}\| < R$ for all $\mathbf{x} \in A$. By Lemma A.5.3, this implies that $\|u_{\mathbf{w}}^2(\mathbf{x}) - u^2(\mathbf{x})\| < \epsilon$ for all $\mathbf{x} \in A$, as required. $\qquad\square$

We may now take a set of m agents, and shift each agent's demand so that its valuation for any bundle is changed by at most ϵ, and nearly all the conditions of Proposition A.4 are met at every intersection of the THs. The only condition we do not insist on is that the set of primitive integer facet normals are unimodular; whether or not this could possibly hold will depend on the demand types of the agents in question. What we prove is that these vectors are linearly independent.

Lemma A.9. *Suppose we have m agents, with valuations u^i for $i = 1, \ldots, m$. For every $\epsilon > 0$ we may perturb each agent's valuation by a vector $\mathbf{w}^i$ such that $\|u^i(\mathbf{x}) - u_{\mathbf{w}^i}^i(\mathbf{x})\| < \epsilon$ for all $\mathbf{x}$ in $\mathbb{R}$, and such that, whenever a price point $\mathbf{p}$ is in the interior of $(n - k_{i_j})$-cell C_{i_j} of the TH $\mathcal{T}_{u^{i_j}}$ for agents $i_1, \ldots, i_s$, then each C_{i_j} is locally to $\mathbf{p}$, given by the intersection of a set of facets $F_1^{i_j}, \ldots, F_{k_{i_j}}^{i_j}$ of $\mathcal{T}_{u^{i_j}}$ (not necessarily comprising all facets of $\mathcal{T}_{u^{i_j}}$ that pass through C_{i_j}) with primitive integer normal vectors $\mathbf{v}_1^{i_j}, \ldots \mathbf{v}_{k_{i_j}}^{i_j}$, such that the full set $\{\mathbf{v}_l^{i_j} \mid j = 1, \ldots, s; l = 1, \ldots, k_{i_j}\}$ is linearly independent.*

Proof. We make a series of perturbations of agents' individual demands, as in Lemmas A.5 and A.8. First, we allow Agent 1 to remain unperturbed. For $i = 2, \ldots, m$ we compare:

1. the TH of aggregate demand of agents $1, \ldots, i - 1$;

2. the TH of agent i.

In each case, we apply Lemma A.8 to find $\mathbf{w}^i$ with $\|u^i(\mathbf{x}) - u_{\mathbf{w}^i}^i(\mathbf{x})\| < \epsilon$, and such that, after the perturbation, $C_i^\circ \cap C^\circ = \{0\}$ whenever $C_i \cap C \neq \emptyset$, where C_i is any cell in $\mathcal{T}_{u^i}$ and C is any cell in the TH of aggregate demand of agents $1, \ldots, i - 1$.

Write U' for the new aggregate demand, after all agents have been perturbed. Now we need to see that the hypotheses of Proposition A.4 are satisfied at every intersection of individual perturbed THs that make up $\mathcal{T}_{U'}$. Consider a price point $\mathbf{p}$, which lies in the interior of $(n - k_{i_j})$-cells $C_{i_1}, \ldots, C_{i_s}$ of the THs of individual demand from distinct agents $i_1, \ldots, i_s$ respectively, where we index so that $i_1 < \cdots < i_s$. From Lemma A.7.3 we know that $\dim C_{i_j}^\circ$ is k_{i_j}.

Let $C := \bigcap_{j=1}^s C_{i_j}$. By Lemma A.7.2, we know that $C^\circ = \left(\bigcap_{j=1}^{s-1} C_{i_j}\right)^\circ + C_{i_s}^\circ$. On the other hand, $\mathbf{p} \in \bigcap_{j=1}^{s-1} C_{i_j}$, and so there is a cell C' of the tropical variety of aggregate demand of agents $1, \ldots, i_s - 1$, with $\mathbf{p} \in C'$. Since demand is constant in the interior of a cell, it follows that $C' \subseteq C_{i_j}$ for $j = 1, \ldots, i_{s-1}$ and so $C' \subseteq \bigcap_{j=1}^{s-1} C_{i_j}$. We know that $\mathbf{p} \in C_{i_s} \cap C'$ and so, by the construction of the perturbations, we know $C'^\circ \cap C_{i_s}^\circ = \{0\}$. As $C' \subseteq \bigcap_{j=1}^{s-1} C_{i_j}$, it follows by Lemma A.7.1 that $\left(\bigcap_{j=1}^{s-1} C_{i_j}\right)^\circ \subseteq C'^\circ$, so we may conclude that $\left(\bigcap_{j=1}^{s-1} C_{i_j}\right)^\circ \cap C_{i_s}^\circ = \{0\}$. Thus

$$C^\circ = \left(\bigcap_{j=1}^{s-1} C_{i_j}\right)^\circ \oplus C_{i_s}^\circ.$$

Proceeding inductively
$$C^\circ = C_{i_1}^\circ \oplus \cdots \oplus C_{i_s}^\circ.$$

We conclude in particular: if $\mathbf{v}_1^{i_j}, \ldots, \mathbf{v}_{k_{i_j}}^{i_j}$ are a basis for $C_{i_j}^\circ$ then $\{\mathbf{v}_l^{i_j} \mid l = 1, \ldots, k_{i_j}; j = 1, \ldots, s\}$ is a set of linearly independent vectors.

But if $C_{i_j} = \bigcap_l F_l^{i_j}$ where $F_l^{i_j}$ are all the facets of this agent's TH of demand which contain C_{i_j} in their boundary, then applying Lemma A.7.2 again, $C_{i_j}^\circ$ is the sum of the spaces $(F_l^{i_j})^\circ$. Each $(F_l^{i_j})^\circ$ is spanned by a single vector $\mathbf{v}_l^{i_j}$, which we may choose to be a primitive integer vector. We may select a maximal linearly independent subset of these vectors, and re-index so these are $\{\mathbf{v}_l^{i_j} \mid l = 1, \ldots, k_{i_j}\}$. Then $C_{i_j}^\circ = \bigoplus_{l=1}^{k_{i_j}} (F_l^{i_j})^\circ$. We already know that $C_{i_j} \subseteq \bigcap_{l=1}^{k_{i_j}} F_l^{i_j}$ so it follows (by Lemma A.7.2) that the affine spans of C_{i_j} and $\bigcap_{l=1}^{k_{i_j}} F_l^{i_j}$ coincide. It follows that C_{i_j} is given, locally around $\mathbf{p}$, by the intersection of the facets $F_1^{i_j}, \ldots, F_{k_{i_j}}^{i_j}$; these facets were chosen above such that their normal vectors are linearly independent. $\qquad\square$

We now have the technical results we need to prove Theorem 6.8.

Proof of Theorem 6.8 Proposition 6.10 covers the case in which condition of the theorem is not satisfied. So suppose that the condition is satisfied. Suppose we have m agents and for $j = 1, \ldots, m$ their valuation is $u^j : A_j \to \mathbb{R}$; write $U : A \to \mathbb{R}$ for the aggregate valuation (as in Section 6.1). We have the tropical variety $\mathcal{T}_U$ of aggregate demand, and the corresponding SNP.

This SNP provides a subdivision of $\text{Conv}(A)$. Our bundle $\mathbf{x}$ may lie at a vertex of the subdivision, in which case there exists a price vector at which it is uniquely demanded. If not, it lies in some k-face of the SNP for some $k \neq 0$. Let $\Delta_{\mathbf{x}}$ be one such k-face. Let $\mathbf{p_x} \in \mathbb{R}^n$ be a price in the corresponding $(n-k)$-cell $C_{\mathbf{x}}$ of aggregate demand. The set $\{\mathbf{y}^\beta \mid \beta \in B\}$ of vertices of $\Delta_{\mathbf{x}}$ are the bundles which are uniquely demanded in an open (n-dimensional) region of $\mathbb{R}^n$ with $C_{\mathbf{x}}$ in its boundary. By assumption there exist $\lambda_\beta \in [0,1]$ with $\sum_\beta \lambda_\beta = 1$ such that $\mathbf{x} = \sum_\beta \lambda_\beta \mathbf{y}^\beta$.

Suppose that $\mathbf{x}$ is not demanded on aggregate at any price. Then, as in the proof of Lemma 2.6, it must follows that $U(\mathbf{x}) < \sum_\beta \lambda_\beta U(\mathbf{y}^\beta)$.

Pick ϵ so that
$$U(\mathbf{x}) < \sum_\beta \lambda_\beta U(\mathbf{y}^\beta) - \epsilon.$$

Now apply Lemma A.9, perturbing agents $j = 2, \ldots, m$ so that their valuation function is altered by no more than $\frac{\epsilon}{3m}$, where we recall that m is the number of agents present. It follows, by assumption regarding the demand type $\mathcal{D}$, that the conditions of Proposition A.4 are satisfied at any intersection of agents' demands. Let U' be the new aggregate demand.

Now $\mathbf{x}$ lies in some k-face of the SNP of this new aggregate demand U', which corresponds to some $(n-k)$-cell of $\mathcal{T}_{U'}$. Let price $\mathbf{p}' \in \mathbb{R}^n$ be in this cell. By Proposition A.4, it follows that $\mathbf{x} \in D_{U'}(\mathbf{p}')$.

However, $\mathbf{x} \in D_{U'}(\mathbf{p}')$ means that $\mathbf{x}$ is weakly preferred on aggregate to any other bundle – including all those in our original vertex set $\{\mathbf{y}^\beta\}$. So, for each $\beta \in B$, we have
$$U'(\mathbf{x}) - \mathbf{x}.\mathbf{p}' \geq U'(\mathbf{y}^\beta) - \mathbf{y}^\beta.\mathbf{p}'. \tag{6}$$

But $U'(\mathbf{x}) = \sum_{j=1}^{m} (u^j)'(\mathbf{x}^j)$, where $\mathbf{x}^j \in A_j$ is the bundle accorded to agent j under this optimal allocation (in particular $\sum_j \mathbf{x}^j = \mathbf{x}$) and $(u^j)'$ is the agent's perturbed valuation function. So

$$\|U'(\mathbf{x}) - U(\mathbf{x})\| = \| \sum_{j=1}^{m}[(u^j)'(\mathbf{x}^j) - u^j(\mathbf{x}^j)]\| \leq \sum_{j=1}^{m} \|(u^j)'(\mathbf{x}^j) - u^j(\mathbf{x}^j)\| \leq m \cdot \frac{\epsilon}{3m} = \frac{\epsilon}{3}$$

and hence $U(\mathbf{x}) + \frac{\epsilon}{3} \geq U'(\mathbf{x})$. Similarly, for all $\beta \in B$, we have $\|U'(\mathbf{y}^\beta) - U(\mathbf{y}^\beta)\| \leq \frac{\epsilon}{3}$ and so $U'(\mathbf{y}^\beta) \geq U(\mathbf{y}^\beta) - \frac{\epsilon}{3}$. Putting these facts together in line (6) we find:

$$U(\mathbf{x}) - \mathbf{x}.\mathbf{p}' \geq U(\mathbf{y}^\beta) - \mathbf{y}^\beta.\mathbf{p}' - \frac{2\epsilon}{3}.$$

Since this holds for all vertices $\mathbf{y}^\beta$ of our original k-face $\Delta_{\mathbf{x}}$ of the SNP, it follows that we may take a weighted sum, using the same weights as originally identified:

$$U(\mathbf{x}) - \mathbf{x}.\mathbf{p}' \geq \sum_{\beta} \lambda_\beta U(\mathbf{y}^\beta) - \sum_{\beta} \lambda_\beta \mathbf{y}^\beta.\mathbf{p}' - \frac{2\epsilon}{3} \implies U(\mathbf{x}) \geq \sum_{\beta} \lambda_\beta U(\mathbf{y}^\beta) - \frac{2\epsilon}{3}.$$

But we originally chose ϵ to satisfy $U(\mathbf{x}) < \sum_{\beta} \lambda_\beta U(\mathbf{y}^\beta) - \epsilon$. This contradiction completes the proof. $\qquad\square$

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