THE EQUILIBRIUM EXISTENCE DUALITY

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Abstract. We show that, with indivisible goods, the existence of competitive equilibrium fundamentally depends on agents' substitution effects, not their income effects. Our Equilibrium Existence Duality allows us to transport results on the existence of competitive equilibrium from settings with transferable utility to settings with income effects. One consequence is that net substitutability—which is a strictly weaker condition than gross substitutability—is sufficient for the existence of competitive equilibrium. Further applications give new existence results beyond the case of (net) substitutes. Our results have implications for auction design.

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1. Introduction

Many markets involve both income effects and the trade of indivisible goods. For example, in high-stakes auctions for blocks of spectrum, financing constraints are often significant even for large telecom companies.

In multi-item auctions with many bidders, competitive equilibrium can be used as an approximation to bidding behavior.[1] However, with indivisible goods, in contrast to the case of divisible goods, competitive equilibrium does not generally exist (Henry, 1970). Moreover, most results about equilibrium existence with indivisible goods assume that utility is transferable—making the problem more tractable,[2] but ruling out income effects. And financing constraints and other income effects make the distribution of wealth among agents affect both Pareto efficiency and aggregate demand, greatly complicating the analysis of competitive equilibrium.

This paper develops a new method to analyze competitive equilibrium when goods are indivisible and there are income effects. We show that whether equilibrium exists is completely determined by substitution effects (i.e., the effects of compensated price changes on agents' demands) as opposed to income or total price effects, and we apply this result to derive new conditions for equilibrium existence.

Our approach is to study exchange economies and competitive equilibrium from the perspective of *Hicksian* demand. As in classical demand theory, Hicksian demand is defined by fixing a utility level and minimizing the expenditure of obtaining it; doing so is dual to the Marshallian approach of fixing an endowment and maximizing utility. Our key methodological innovation is to combine Hicksian demands to construct a family of "Hicksian economies" in each of which prices vary, but agents' utilities—rather than their endowments—are held constant. Our main result, the "Equilibrium Existence Duality," states that competitive equilibria exist for all (feasible) profiles of endowments in the original economy if and only if competitive equilibria exist in the Hicksian economies for all utility levels.

Our Equilibrium Existence Duality has conceptual and technical consequences for understanding when equilibrium exists. To understand these consequences, note that preferences in each Hicksian economy reflect agents' substitution effects. Therefore, by the Equilibrium

[1] See, for example, Klemperer (2008, 2010) and Milgrom (2008, 2009).

[2] For example, the assumption that utility is transferable allows the set of Pareto-efficient allocations to be characterized in terms of a welfare maximization problem, enabling the analysis of competitive equilibrium using methods based on integer programming (see, e.g., Koopmans and Beckmann (1957), Bikhchandani and Mamer (1997), Ma (1998), Candogan et al. (2015), and Tran and Yu (2019)). The assumption also allows aggregate demand to be represented as the demand of a representative agent, enabling the analysis of competitive equilibrium using methods based on convex programming (see, e.g., Murota (2003), Ikebe et al. (2015), and Candogan, Epitropou, and Vohra (2021)) and tropical geometry (Baldwin and Klemperer, 2019).

Existence Duality, the existence of competitive equilibrium fundamentally depends on substitution effects. More precisely, it follows from the Equilibrium Existence Duality that for any condition that guarantees the existence of equilibrium, either that condition, or a weaker sufficient condition, must be a condition on substitution effects alone. Moreover, as fixing a utility level precludes income effects, agents' preferences in each Hicksian economy have quasilinear representations. Hence, the Equilibrium Existence Duality allows us to transport (and so generalize) *any* necessary or sufficient condition for equilibrium existence from settings with transferable utility to settings with income effects.

For example, suppose that each agent demands at most one unit of each good. In this case, with transferable utility (so no income effects), substitutability is sufficient for the existence of competitive equilibrium (Kelso and Crawford, 1982) and defines a maximal domain for existence (Gul and Stacchetti, 1999). The Equilibrium Existence Duality therefore implies that, with income effects, *net* substitutability—a condition on substitution effects—is both sufficient for, and defines a maximal domain for, the existence of competitive equilibrium.

Previous work focused instead on *gross* substitutability—a condition on total price effects that is equivalent to net substitutability when there are no income effects. For example, Fleiner et al. (2019) built on arguments of Kelso and Crawford (1982) to show that, with income effects, competitive equilibrium exists under gross substitutability. But given that substitution effects fundamentally determine whether equilibrium exists, there must be a weaker sufficient condition that depends only on substitution effects. And indeed we show that, with income effects, gross substitutability implies net substitutability, but not vice versa. Thus, an implication of our results is that it is unfortunate that Kelso and Crawford (1982), and much of the subsequent literature, used the term "gross substitutes" to refer to a substitutability condition on quasilinear utility functions when discussing equilibrium existence—our work shows that it is net substitutability, not gross substitutability, that is critical to equilibrium existence with substitutes.[3]

To appreciate the distinction between gross and net substitutability, suppose that Martine is thinking about selling her house and buying another. If the price of her own house increases and she is financially constrained, then she may wish to buy a luxurious house instead of a spartan one—exposing a gross complementarity between her existing house and the spartan one. However, Martine regards the houses as net substitutes: the complementarity emerges entirely due to an income effect.[4] Competitive equilibrium is therefore guaranteed to exist

[3]Kelso and Crawford (1982) noted the equivalence between gross and net substitutability in their setting (see their Footnote 1) but used the term "gross substitutes" to analogize their arguments for existence with tâtonnement from general equilibrium theory. Their model allows for income effects for the (unit-supply) workers but not for the (multi-unit demand) firms.

[4]Similar logic applies to agents who are not endowed with any goods: in this case, raising the price of a good that an agent is demanding makes the agent poorer, which can lead to gross complementarities even when the agent regards goods as net substitutes.

in economies with Martine if all other agents see the goods as net substitutes (and each demand at most one unit of each good), despite the presence of gross complementarities.

The Equilibrium Existence Duality leads to new existence results beyond the case of net substitutes. For example, it leads to an extension of Bikhchandani and Mamer's (1997) necessary and sufficient condition for the existence of equilibrium to settings with income effects. It also yields a new existence result for matching models under a net complementarity condition—extending a result of Rostek and Yoder (2020b).

Our analysis may have significant implications for the design of auctions that seek competitive equilibrium outcomes, and in which bidders face financing constraints. In addition to existence, gross substitutability has also been used to show that ascending auctions converge to competitive equilibrium under straightforward bidding.[5] However, as we illustrate, even under net substitutability, gross complementarities do prevent dynamic auctions from monotonically finding a competitive equilibrium, because increasing the price of an over-demanded good may lead to another good becoming under-demanded. Nevertheless, our results suggest that sealed-bid auction designs, such as versions of the Product-Mix Auction (Klemperer, 2010) used by the Bank of England since the Global Financial Crisis, may work well in this context.

Several other papers have considered the existence of competitive equilibrium in the presence of indivisibilities and income effects. Quinzii (1984), Gale (1984), and Svensson (1984) showed the existence of competitive equilibrium in a housing market economy in which agents have unit demand and endowments. Building on those results, Kaneko and Yamamoto (1986), van der Laan et al. (1997, 2002), and Yang (2000) analyzed settings with multiple goods, but restricted attention to separable preferences. By contrast, our results allow for interactions between the demand for different goods. We also clarify the role of net substitutability for the existence of competitive equilibrium.

In a different direction, the important work of Danilov et al. (2001) implies a version of our existence result for net substitutes. There are three key differences between the current paper and Danilov et al. (2001). First, Danilov et al.'s (2001) work did not reveal the crucial role of substitution effects in determining the existence of equilibrium. For example, the sufficiency of net substitutability for the existence of equilibrium has not been previously observed. Second, Danilov et al. (2001) focused on sufficient conditions, while our approach also yields maximal domain results. Finally, our approach allows us to port *any* (necessary or sufficient) condition for the existence of equilibrium in transferable utility to settings with income effects—leading, as we show in Section 5, to results beyond the classes of preferences that Danilov et al.'s (2001) methods can cover. We discuss the relationship in detail in Section 4.5.

[5]See, for example, Kelso and Crawford (1982), Gul and Stacchetti (2000), Milgrom (2000), and Fleiner et al. (2019).

Subsequent to our original work, Nguyen and Vohra (2022) have refined some of Danilov et al. (2001) ideas and arguments to prove another version of our existence result for net substitutes, and also to provide results on the existence of approximate equilibrium in the presence of complementarities. By studying the structure of the Marshallian demand set at prices at which demand is non-unique, they formulated a "geometric substitutes" condition that ensures the existence of competitive equilibrium for specific money endowments. However, the geometric substitutes condition depends on money endowments; requiring it for all money endowments is equivalent to assuming net substitutability.

From a technical perspective, our paper is related to Luenberger's (1994) proof of the standard equilibrium existence result for settings with divisible goods and convex preferences. Luenberger's (1994) argument proceeds by relating competitive equilibrium to the solutions to a family of optimization problems, which turn out to characterize competitive equilibrium prices in our Hicksian economies, and then applying a topological fixed-point argument. It turns out that these steps in his argument do not rely on convexity. However, the connection to substitution effects and the equilibrium existence problem with indivisible goods has not been previously made. We discuss the differences between Luenberger's (1994) proof strategy and our arguments in Section 3.

We proceed as follows. Section 2 describes our setting—an exchange economy with indivisible goods and money. Section 3 develops the Equilibrium Existence Duality. Section 4 specializes to the case of substitutes. Section 5 develops further applications of the Equilibrium Existence Duality beyond the case of substitutes, and Section 6 is a conclusion. The appendices provide proofs.

2. The Setting

We work with a model of exchange economies with indivisibilities—adapted to allow for income effects. There is a finite set J of agents, a finite set I of indivisible goods, and a divisible numéraire that we call "money." We allow goods to be undesirable, i.e., to be "bads." We fix a *total endowment of goods* in the economy, which we denote by $\mathbf{y}_I \in \mathbb{Z}^I$.[6]

2.1. Preferences and Marshallian Demand.

Each agent $j \in J$ has a finite set $X^j_I \subseteq \mathbb{Z}^I$ of *feasible bundles* of indivisible goods and a lower bound $\underline{x}^j_0 \geq -\infty$ on her consumption of money. As bundles that specify negative consumption of some goods can be feasible, our setting implicitly allows for production.[7] The principal cases of $\underline{x}^j_0$ are $\underline{x}^j_0 = -\infty$, in which case all levels of consumption of money are feasible, and $\underline{x}^j_0 = 0$, in which case the

[6] In particular, we allow for multiple units of some goods to be present in the aggregate, unlike Gul and Stacchetti (1999) and Candogan et al. (2015).

[7] Technological constraints on production (in the sense of Hatfield et al. (2013) and Fleiner et al. (2019)) can be represented by the possibility that some bundles of goods are infeasible for an agent to consume (see Example 2.15 in Baldwin and Klemperer (2014)).

consumption of money must be positive. Hence, the set of feasible consumption bundles for agent j is $X^j = (\underline{x}_0^j, \infty) \times X_I^j$. Given a bundle $\mathbf{x} \in X^j$, we let x_0 denote the amount of money in $\mathbf{x}$ and $\mathbf{x}_I$ denote the bundle of goods specified by $\mathbf{x}$, so $\mathbf{x} = (x_0, \mathbf{x}_I)$.

The utility levels of agent j lie in the range $(\underline{u}^j, \overline{u}^j)$, where $-\infty \leq \underline{u}^j < \overline{u}^j \leq \infty$. Furthermore, each agent j has a *utility function* $U^j : X^j \to (\underline{u}^j, \overline{u}^j)$ that we assume to be continuous and strictly increasing in x_0, and to satisfy

$$\lim_{x_0 \to (\underline{x}_0^j)^+} U^j(x_0, \mathbf{x}_I) = \underline{u}^j \quad \text{and} \quad \lim_{x_0 \to \infty} U^j(x_0, \mathbf{x}_I) = \overline{u}^j \tag{1}$$

for all $\mathbf{x}_I \in X_I^j$. Condition (1) requires that some consumption of money above the minimum level $\underline{x}_0^j$ be essential to agent j.[8] If Condition (1) is not satisfied, then it is known that competitive equilibrium may not exist (Mas-Colell, 1977)—even in settings in which agents have unit demand for goods (see, e.g., Herings and Zhou (2022)). However, Condition (1) requires that agents who cannot go into debt be unwilling to spend their entire money endowment—ruling out the possibility that they face hard budget constraints.[9]

Given an endowment $\mathbf{w} = (w_0, \mathbf{w}_I) \in \mathbb{R} \times \mathbb{Z}^I$ of money and goods, and a price vector $\mathbf{p}_I \in \mathbb{R}^I$, agent j's *Marshallian demand* for goods is

$$D_{\mathrm{M}}^j(\mathbf{p}_I, \mathbf{w}) = \left\{ \mathbf{x}_I^* \;\middle|\; \mathbf{x}^* \in \operatorname*{arg\,max}_{\mathbf{x} \in X^j | \mathbf{p} \cdot \mathbf{x} \leq \mathbf{p} \cdot \mathbf{w}} U^j(\mathbf{x}) \right\}.$$

Here, $p_0 = 1$ since money is the numéraire. As usual, Marshallian demand is given by the set of bundles of goods that maximize an agent's utility, subject to a budget constraint, given a price vector and an endowment. While $D_{\mathrm{M}}^j(\mathbf{p}_I, \mathbf{w})$ may be empty if no feasible consumption bundle is affordable, it is non-empty if (but not only if) $\mathbf{w} \in X^j$. An *income effect* is a change in an agent's Marshallian demand induced by a change in her money endowment, holding prices fixed.

Our setup can capture the standard quasilinear setting. Specifically, given a *valuation* $V^j : X_I^j \to \mathbb{R}$, letting $\underline{x}_0^j = \underline{u}^j = -\infty$ and $\overline{u}^j = \infty$, one obtains a quasilinear utility function given by

$$U^j(x_0, \mathbf{x}_I) = x_0 + V^j(\mathbf{x}_I).$$

When agents' utility functions are quasilinear, they do not experience income effects, so Marshallian demand coincides with *demand*

$$D^j(\mathbf{p}_I) = \operatorname*{arg\,max}_{\mathbf{x}_I \in X_I^j} \{V^j(\mathbf{x}_I) - \mathbf{p}_I \cdot \mathbf{x}_I\}. \tag{2}$$

[8]Henry (1970, pages 543–544), Mas-Colell (1977, Theorem 1(i)), and Demange and Gale (1985, Equation (3.1)) made similar assumptions.

[9]Gul et al. (2020) and Jagadeesan and Teytelboym (2021) have addressed the existence problem for settings with (hard) budget constraints by assuming that agents can trade lotteries over goods, and by considering stable matchings, respectively.

When all agents have quasilinear utility functions, we say that utility is *transferable.*

2.2. **Hicksian Demand and Hicksian Valuations.** Given a utility level $u \in (\underline{u}^j, \overline{u}^j)$ and a price vector $\mathbf{p}_I$, agent j's *Hicksian demand* for goods is

$$D_{\mathrm{H}}^j(\mathbf{p}_I; u) = \left\{ \mathbf{x}_I^* \,\middle|\, \mathbf{x}^* \in \underset{\mathbf{x} \in X^j | U^j(\mathbf{x}) \geq u}{\arg\min} \mathbf{p} \cdot \mathbf{x} \right\}. \tag{3}$$

As in the standard divisible goods case, Hicksian demand is given by the set of bundles of goods that minimize the expenditure of obtaining a utility level given a price vector. A *substitution effect* is a change in an agent's Hicksian demand induced by a change in prices, holding her utility level fixed.

As in classical demand theory, a bundle of goods is expenditure-minimizing if and only if it is utility-maximizing.[10]

Fact 1 (Relationship between Marshallian and Hicksian Demand)**.** *Let $\mathbf{p}_I$ be a price vector.*

(a) For all endowments $\mathbf{w}$ with $D_{\mathrm{M}}^j(\mathbf{p}_I, \mathbf{w}) \neq \varnothing$, we have that $D_{\mathrm{M}}^j(\mathbf{p}_I, \mathbf{w}) = D_{\mathrm{H}}^j(\mathbf{p}_I; u)$, where

$$u = \max_{\mathbf{x} \in X^j | \mathbf{p} \cdot \mathbf{x} \leq \mathbf{p} \cdot \mathbf{w}} U^j(\mathbf{x}).$$

(b) For all utility levels u and endowments $\mathbf{w}$ with

$$\mathbf{p} \cdot \mathbf{w} = \min_{\mathbf{x} \in X^j | U^j(\mathbf{x}) \geq u} \mathbf{p} \cdot \mathbf{x},$$

we have that $D_{\mathrm{H}}^j(\mathbf{p}_I; u) = D_{\mathrm{M}}^j(\mathbf{p}_I, \mathbf{w})$.

If an agent has a quasilinear utility function, then her Hicksian demand coincides with her demand (i.e., $D_{\mathrm{H}}^j(\mathbf{p}_I; u) = D^j(\mathbf{p}_I)$). We next show that the representation of the expenditure minimization problem as a quasilinear maximization problem persists in the presence of income effects, by using the constraint in Equation (3) to solve for x_0 as a function of $\mathbf{x}_I$.

Definition 1. The *Hicksian valuation* $V_{\mathrm{H}}^j(\cdot; u) : X_I^j \to \mathbb{R}$ of agent j at utility level u is given by

$$V_{\mathrm{H}}^j(\mathbf{x}_I; u) = -U^j(\cdot, \mathbf{x}_I)^{-1}(u).$$

[10] Although Fact 1 is usually stated with divisible goods (see, e.g., Proposition 3.E.1 and Equation (3.E.4) in Mas-Colell et al. (1995)), the standard proof applies with multiple indivisible goods and money under Condition (1). For sake of completeness, we give a proof of Fact 1 in Appendix C. Without Condition (1), the conclusion of Fact 1(b) can fail even with strictly increasing utility functions (Jagadeesan and Teytelboym, 2021).

Thus, we have that $U^j\left(-V^j_{\mathrm{H}}(\mathbf{x}_I;u),\mathbf{x}_I\right)=u$; the minus sign serves to convert (expenditure) minimization to (quasilinear) maximization.[11] And agent j's Hicksian demand at utility level u is the demand correspondence of an agent with valuation $V^j_{\mathrm{H}}(\cdot;u)$.

Lemma 1. *For all price vectors $\mathbf{p}_I$ and utility levels u, we have that*

$$D^j_{\mathrm{H}}(\mathbf{p}_I;u)=\operatorname*{arg\,max}_{\mathbf{x}_I\in X^j_I}\left\{V^j_{\mathrm{H}}(\mathbf{x}_I;u)-\mathbf{p}_I\cdot\mathbf{x}_I\right\}.$$

Proof. As $U^j(\mathbf{x})$ is strictly increasing in x_0, we have that

$$D^j_{\mathrm{H}}(\mathbf{p}_I;u)=\left\{\mathbf{x}^*_I\;\middle|\;\mathbf{x}^*\in\operatorname*{arg\,min}_{\mathbf{x}\in X^j|U^j(\mathbf{x})=u}\mathbf{p}\cdot\mathbf{x}\right\}=\left\{\mathbf{x}^*_I\;\middle|\;\mathbf{x}^*\in\operatorname*{arg\,max}_{\mathbf{x}\in X^j|U^j(\mathbf{x})=u}\{-\mathbf{p}\cdot\mathbf{x}\}\right\}.$$

Applying the substitution $x_0=-V^j_{\mathrm{H}}(\mathbf{x}_I;u)$ to remove the constraint from the minimization problem yields the lemma. □

It follows from Lemma 1 that an agent's Hicksian valuation at a utility level gives rise to a quasilinear utility function that reflects the agent's substitution effects at that utility level. Lemma 1 also yields a relationship between the family of Hicksian valuations and income effects. Indeed, by Fact 1, an agent's income effects correspond to changes in her Hicksian demand induced by changes in her utility level, holding prices fixed.[12]

2.3. **The Hicksian Economies.** We combine the families of Hicksian valuations to form a family of Hicksian economies, in each of which utility is transferable and agents choose consumption bundles to minimize the expenditure of obtaining given utility levels.

Definition 2. The *Hicksian economy for a profile of utility levels* $(u^j)_{j\in J}$ is the transferable utility economy in which agent j's valuation is $V^j_{\mathrm{H}}(\cdot;u^j)$.

The family of Hicksian economies consists of the "duals" of the original economy in which income effects have been removed and price effects are given by substitution effects. Importantly, the construction of the Hicksian economies allows us to convert economies with income effects to families of economies with transferable utility.

To illustrate, consider Quinzii's (1984) housing market model. In that setting, each Hicksian economy is an assignment game in the sense of Koopmans and Beckmann (1957)—illustrating a connection between these two classic models.

Example 1 (A Housing Market—Quinzii, 1984; Gale, 1984; Svensson, 1984)**.** For each agent j, let $X^j_I\subseteq\{\mathbf{0}\}\cup\{\mathbf{e}^i\mid i\in I\}$ be nonempty. Each agent is endowed with an indivisible good,

[11]The function $-V^j_{\mathrm{H}}$ is the *compensation function* of Demange and Gale (1985) (see also Danilov et al. (2001)). The Hicksian valuation is a variant of the *benefit function* of Luenberger (1992a) for settings with indivisible goods and a fixed numéraire.

[12]It therefore also follows from Lemma 1 that agent j's preferences exhibit income effects if and only if $V^j_{\mathrm{H}}(\mathbf{x}_I;u)$ is not additively separable between $\mathbf{x}_I$ and u.

so the total endowment of goods $\mathbf{y}_I$ consists of $|J|$ units of goods. In each Hicksian economy, utility is transferable and agents have unit demand for the goods. As endowments are irrelevant in transferable utility economies, each Hicksian economy is an assignment game in the sense of Koopmans and Beckmann (1957) and Shapley and Shubik (1971).

3. The Equilibrium Existence Duality

We now turn to the analysis of competitive equilibrium in exchange economies. An *endowment profile* consists of an endowment $\mathbf{w}^j$ for each agent j. An endowment profile is *feasible* if $\mathbf{w}^j \in X^j$ for all agents j, and $\sum_{j \in J} \mathbf{w}_I^j = \mathbf{y}_I$, where $\mathbf{y}_I$ is the total endowment of goods. Given a feasible endowment profile, a competitive equilibrium specifies a price vector such that markets for goods clear when agents maximize utility. By Walras's Law, it follows that the market for money clears as well.

Definition 3. Given a feasible endowment profile $(\mathbf{w}^j)_{j \in J}$, a *competitive equilibrium* consists of a price vector $\mathbf{p}_I$ and a bundle $\mathbf{x}_I^j \in D_{\mathrm{M}}^j(\mathbf{p}_I, \mathbf{w}^j)$ for each agent such that $\sum_{j \in J} \mathbf{x}_I^j = \mathbf{y}_I$.

In transferable utility economies, a competitive equilibrium consists of a price vector $\mathbf{p}_I$ and a bundle $\mathbf{x}_I^j \in D^j(\mathbf{p}_I)$ for each agent such that $\sum_{j \in J} \mathbf{x}_I^j = \mathbf{y}_I$. In this case, the feasible endowment profile does not affect competitive equilibrium because endowments do not affect (Marshallian) demand. We therefore omit the (feasible) endowment profile when considering competitive equilibrium in transferable utility economies in which a feasible endowment profile exists—i.e., $\mathbf{y}_I \in \sum_{j \in J} X_I^j$. On the other hand, the total endowment of goods $\mathbf{y}_I$ affects competitive equilibrium even when utility is transferable.

Recall that utility is transferable in the Hicksian economies. Furthermore, by Lemma 1, a competitive equilibrium in the Hicksian economy for a profile $(u^j)_{j \in J}$ of utility levels consists of a price vector $\mathbf{p}_I$ and a bundle $\mathbf{x}_I^j \in D_{\mathrm{H}}^j(\mathbf{p}_I; u^j)$ for each agent such that $\sum_{j \in J} \mathbf{x}_I^j = \mathbf{y}_I$. Thus, agents act as if they minimize expenditure in competitive equilibrium in the Hicksian economies.[13]

Our Equilibrium Existence Duality connects the equilibrium existence problems in the original economy (which can feature income effects) and the Hicksian economies (in which utility is transferable). Specifically, we show that competitive equilibrium always exists in the original economy if and only if it always exists in the Hicksian economies. Here, we

[13] As a result, competitive equilibria in the Hicksian economies coincide with *quasiequilibria with transfers* from the modern treatment of the Second Fundamental Theorem of Welfare Economics (see, e.g., Definition 16.D.1 in Mas-Colell et al. (1995)). As the set of feasible levels of money consumption is open, agents always can always reduce their money consumption slightly from a feasible bundle to obtain a strictly cheaper feasible bundle. Hence, quasiequilibria with transfers coincide with equilibria with transfers in the original economy (see, e.g., Proposition 16.D.2 in Mas-Colell et al. (1995) for the case of divisible goods). If the endowments of money were fixed in the Hicksian economies, this concept would coincide with the concept of *compensated equilibrium* of Arrow and Hahn (1971) and the concept of *quasiequilibrium* introduced by Debreu (1962).

hold agents' preferences and the total endowment of goods fixed, but allow the endowment profile—and hence the total endowment of money—to vary.

Theorem 1 (Equilibrium Existence Duality)**.** *Suppose that a feasible endowment profile exists. Competitive equilibria exist for all feasible endowment profiles if and only if competitive equilibria exist in the Hicksian economies for all profiles of utility levels.*

The Equilibrium Existence Duality has a conceptual implication for the roles of substitution effects vis-à-vis income effects in determining whether competitive equilibrium exists. Indeed, by Lemma 1, agents' substitution effects determine their preferences in each Hicksian economy. Therefore, the Equilibrium Existence Duality tells us that for *any* condition that ensures the existence of competitive equilibria for all feasible endowment profiles, either that condition or a weaker sufficient condition can be written as a condition on substitution effects alone.[14] For example, since the "gross substitutability" condition—a condition on total price effects—is sufficient for the existence of competitive equilibrium in the presence of income effects, there must be a weaker sufficient condition that can be expressed in terms of substitution effects alone—namely our "net substitutability" condition (see Section 4.3). Moreover, any weakest condition for equilibrium existence can be written as a condition on substitution effects alone. In this sense, substitution effects fundamentally determine whether competitive equilibrium exists.

Both directions of the Equilibrium Existence Duality also lead to new technical results on when competitive equilibrium exists. As demands in the Hicksian economies are given by Hicksian demand in the original economy (Lemma 1), the "if" direction of Theorem 1 implies that every condition on demand D^j that guarantees the existence of competitive equilibrium in settings with transferable utility translates into a condition on Hicksian demand D^j_{H} that guarantees the existence of competitive equilibrium in settings with income effects. In Sections 4 and 5, we use the "if" direction of Theorem 1 to obtain new sufficient conditions for the existence of competitive equilibrium with income effects from previous equilibrium existence results for settings with transferable utility (Kelso and Crawford, 1982; Bikhchandani and Mamer, 1997; Rostek and Yoder, 2020b). Conversely, the "only if" direction of Theorem 1 shows that if a condition on demand is necessary for (resp. defines a maximal domain for) equilibrium existence in settings with transferable utility, then the translated condition on Hicksian demand is necessary for (resp. defines a maximal domain for) the existence of competitive equilibrium in settings with income effects. In Sections 4 and 5, we also use this implication to derive new maximal domain results and necessity results for settings with income effects.

[14]Similarly, the Equilibrium Existence Duality tells us that any condition that is necessary for the existence of competitive equilibrium for all feasible endowment profiles, either that condition or a stronger necessary condition can be expressed as a condition on substitution effects alone.

To prove the "only if" direction of Theorem 1, we exploit a version of the Second Fundamental Theorem of Welfare Economics for settings with indivisibilities. To understand connection to the existence problem for the Hicksian economies, note that the existence of competitive equilibrium in the Hicksian economies is equivalent to the conclusion of the Second Welfare Theorem—i.e., that each Pareto-efficient allocation can be supported in an equilibrium with endowment transfers—as the following lemma shows.[15]

Lemma 2. *Suppose that a feasible endowment profile exists. Competitive equilibria exist in the Hicksian economies for all profiles of utility levels if and only if, for each Pareto-efficient allocation* $(\mathbf{x}^j)_{j\in J}$ *with* $\sum_{j\in J}\mathbf{x}^j_I = \mathbf{y}_I$*, there exists a price vector* $\mathbf{p}_I$ *such that* $\mathbf{x}^j \in D^j_{\mathrm{M}}(\mathbf{p}_I, \mathbf{x}^j)$ *for all agents* j*.*

We prove Lemma 2 in Appendix A. Intuitively, as utility is transferable in the Hicksian economies, variation in utility levels between Hicksian economies plays that same role as endowment transfers in the Second Welfare Theorem. It is well-known that the conclusion of the Second Welfare Theorem holds whenever competitive equilibria exist for all feasible endowment profiles (Maskin and Roberts, 2008).[16] It follows that competitive equilibrium always exists in the Hicksian economies whenever it always exists in the original economy, which is the "only if" direction of Theorem 1.

We use a different argument to prove the "if" direction. Our strategy is to show that there exists a profile of utility levels and a competitive equilibrium in the corresponding Hicksian economy in which all agents' expenditures equal their budgets in the original economy. To do so, we apply a topological fixed-point argument. We consider an auctioneer who, for a given profile of candidate equilibrium utility levels, evaluates agents' expenditures over all competitive equilibria in the Hicksian economy and adjusts candidate equilibrium utility levels upwards (resp. downwards) for agents who under- (resp. over-) spend their budgets. The existence of competitive equilibrium in the Hicksian economies ensures that the process is nonempty-valued, and the transferability of utility in the Hicksian economies ensures that the process is convex-valued. Kakutani's Fixed Point Theorem implies the existence of a fixed-point utility profile. By construction, there exists a competitive equilibrium in the corresponding Hicksian economy at which agents' expenditures equal the values of their

[15]Recall that an allocation $(\mathbf{x}^j)_{j\in J} \in \bigtimes_{j\in J} X^j$ is *Pareto-efficient* if there does not exist an allocation $(\hat{\mathbf{x}}^j)_{j\in J} \in \bigtimes_{j\in J} X^j$ such that

$$\sum_{j\in J}\hat{\mathbf{x}}^j = \sum_{j\in J}\mathbf{x}^j,$$

and $U^j\left(\hat{\mathbf{x}}^j\right) \geq U^j\left(\mathbf{x}^j\right)$ for all agents j with strict inequality for some agent.

[16]While Maskin and Roberts (2008) assumed that goods are divisible, their arguments apply even in the presence of indivisibilities—as we show in Appendix A.

endowments. By Fact 1 and Lemma 1, agents must be maximizing utility given their endowments at this equilibrium, and hence once obtains a competitive equilibrium in the original economy. The details of the argument are in Appendix A.

This argument is closely related to Luenberger's (1994) proof of the standard equilibrium existence result for settings with divisible goods and convex preferences. Luenberger's (1994) argument is based on adjusting prices to simultaneously minimize total surplus and achieve utility levels that make expenditures equal incomes. In our proof of the "if" direction of Theorem 1, instead of considering prices that minimize surplus, we search over competitive equilibria in the Hicksian economies.[17] Furthermore, we apply a topological fixed-point argument based on adjusting candidate equilibrium utility levels, rather than prices.[18] This approach has the advantage of working even when competitive equilibrium prices can be unbounded, such as in the presence of technological constraints (in the sense of Hatfield et al. (2013) and Fleiner et al. (2019)).

4. The Case of Substitutes

In this section, we apply the Equilibrium Existence Duality to prove a new result regarding the existence of competitive equilibrium with substitutable indivisible goods and income effects: we show that a form of *net* substitutability is sufficient for, and defines a maximal domain for, the existence of competitive equilibrium. We begin by reviewing previous results on the existence of competitive equilibrium with substitutable indivisible goods in transferable utility economies. We then derive our existence theorem for net substitutability, and relate it to previous results that assume gross substitutability. Last, we derive a maximal domain result for net substitutability.

We focus on the case in which each agent demands at most one unit of each good. Formally, we say that an agent j *demands at most one unit of each good* if $X_I^j \subseteq \{0,1\}^I$. We can capture the possibility that agents demand more than one unit of a commodity by expanding the set of goods to include multiple goods corresponding to units of a single commodity (see, e.g., Bikhchandani and Mamer (1997), Gul and Stacchetti (1999), and Milgrom and Strulovici (2009)). Although this transformation would allow different units of the same commodity to have different prices, if competitive equilibrium exists after applying this transformation,

[17]By applying duality characterizations of competitive equilibrium in transferable utility economies (as used, e.g., in the analysis of Bikhchandani and Mamer (1997)) to each Hicksian economy, one can show that these approaches are equivalent when competitive equilibrium exist in each Hicksian economy—which is precisely the hypothesis of the "if" direction of Theorem 1.

[18]This approach is similar in spirit to Negishi's (1960) proof of the existence of competitive equilibrium with divisible goods. Negishi (1960) instead applied an adjustment process to the inverses of agents' marginal utilities of money. However, Negishi's (1960) approach does not generally yield a convex-valued adjustment process in the presence of indivisibilities.

then competitive equilibrium exists in which goods corresponding to different units of the same commodity have the same price.[19]

4.1. **Substitutability and the Existence of Competitive Equilibrium in Transferable Utility Economies.** Substitutability requires that increases in the price of a good weakly raise demand for all other goods. Kelso and Crawford (1982) called this condition "gross substitutability," but with quasilinear utility, the modifier "gross" can be dropped—as in classical demand theory.

Definition 4 (Substitutability—Kelso and Crawford, 1982; Ausubel and Milgrom, 2002)**.** A valuation V^j is a *substitutes valuation* if for all price vectors $\mathbf{p}_I$ and $\lambda > 0$, whenever $D^j(\mathbf{p}_I) = \{\mathbf{x}_I\}$ and $D^j(\mathbf{p}_I + \lambda\mathbf{e}^i) = \{\mathbf{x}'_I\}$, we have that $x'_k \geq x_k$ for all goods $k \neq i$.[20]

It is well-known that when utility is transferable and each agent demands at most one unit of each good, competitive equilibrium exists under substitutability.

Fact 2. *Suppose that utility is transferable and that a feasible endowment profile exists. If each agent demands at most one unit of each good and has a substitutes valuation, then competitive equilibrium exists.*[21]

4.2. **Net Substitutability and the Existence of Competitive Equilibrium.** In light of Fact 2 and the Equilibrium Existence Duality, competitive equilibrium exists if agents' Hicksian demands satisfy an appropriate substitutability condition—i.e., if preferences satisfy a *net* analogue of substitutability.

We build on Definition 4 to define a version of the net substitutability property from classical consumer theory for settings with indivisibilities. The condition requires that *compensated* increases in the price of a good (i.e., price increases that are offset by compensating transfers) weakly raise demand for all other goods.

Definition 5 (Net Substitutability)**.** A utility function U^j is a *net substitutes utility function* if for all utility levels u, price vectors $\mathbf{p}_I$, and $\lambda > 0$, whenever $D^j_{\mathrm{H}}(\mathbf{p}_I; u) = \{\mathbf{x}_I\}$ and $D^j_{\mathrm{H}}(\mathbf{p}_I + \lambda\mathbf{e}^i; u) = \{\mathbf{x}'_I\}$, we have that $x'_k \geq x_k$ for all goods $k \neq i$.

[19]See, for example, Bikhchandani and Mamer (1997, p. 391). See Baldwin, Edhan, Jagadeesan, Klemperer, and Teytelboym (2021) for details of the extension of the results of this section to settings in which agents can demand multiple units of each good.

[20]Kelso and Crawford (1982) imposed a (gross) substitutability condition at all price vectors. Our definition of substitutability, which is due to Ausubel and Milgrom (2002), considers only price vectors at which demand is single-valued, but coincides with Kelso and Crawford's (1982) definition when at most one unit of each good is demanded (Danilov, Koshevoy, and Lang, 2003; Shioura and Tamura, 2015; Hatfield et al., 2019).

[21]Fact 2 is a version of Theorem 1 in Hatfield et al. (2013) for exchange economies and follows from Proposition 4.6 in Baldwin and Klemperer (2019). See Kelso and Crawford (1982) and Gul and Stacchetti (1999) for earlier versions that assume that valuations are monotone.

For quasilinear utility functions, as Hicksian demand is independent of the utility level and coincides with demand, net substitutability is equivalent to substitutability. More generally, net substitutability can be expressed as a condition on Hicksian valuations.

Remark 1. By Lemma 1, if an agent demands at most one unit of each good, then she has a net substitutes utility function if and only if her Hicksian valuations at all utility levels are substitutes valuations. As a result, alternative characterizations of substitutes valuations (such as those of Gul and Stacchetti (1999), Ausubel and Milgrom (2002), Fujishige and Yang (2003), Shioura and Tamura (2015), and Hatfield et al. (2019)), when applied to the Hicksian valuations, yield characterizations of net substitutes utility functions as well.

Corollary 1. *If all agents demand at most one unit of each good and have net substitutes utility functions, then competitive equilibria exist for all feasible endowment profiles.*

Corollary 1 is an immediate consequence of the Equilibrium Existence Duality and the existence of competitive equilibria in transferable utility economies under substitutability.

Proof. Remark 1 implies that the agents' Hicksian valuations at all utility levels are substitutes valuations. Hence, Fact 2 implies that competitive equilibria exist in the Hicksian economies for all profiles of utility levels if a feasible endowment profile exists. The theorem follows by the "if" direction of Theorem 1. $\square$

4.3. **Net Substitutability versus Gross Substitutability.** Previous work has shown that competitive equilibrium exists—and can be found or approximated by monotone, dynamic auctions—under a condition called gross substitutability. Gross substitutability is a version of the gross substitutability condition from classical consumer theory and requires that *uncompensated* increases in the price of a good weakly raise demand for all other goods.

Definition 6 (Gross Substitutability–Kelso and Crawford, 1982; Fleiner et al., 2019)**.** A utility function U^j is a *gross substitutes utility function at endowment* $\mathbf{w}$ if for all price vectors $\mathbf{p}_I$, and $\lambda > 0$, whenever $D^j_{\mathrm{M}}(\mathbf{p}_I, \mathbf{w}) = \{\mathbf{x}_I\}$ and $D^j_{\mathrm{M}}(\mathbf{p}_I + \lambda \mathbf{e}^i, \mathbf{w}) = \{\mathbf{x}'_I\}$, we have that $x'_k \geq x_k$ for all goods $k \neq i$.[22]

Generalizing Fact 2 to settings with income effects, Fleiner et al. (2019) showed that competitive equilibrium exists for a feasible endowment profile $(\mathbf{w}^j)_{j \in J}$ if each agent j's utility function is a gross substitutes utility function at her endowment $\mathbf{w}^j$.[23]

[22]Our definition is analogous to the "full substitutability in demand language" condition from Assumption D.1 in Supplemental Appendix D of Fleiner et al. (2019). Kelso and Crawford (1982) imposed a gross substitutability condition at all price vectors—instead of only price vectors at which demand is single-valued—which leads to a strictly stronger condition in the presence of income effects (Schlegel, 2022), even when at most one unit of each good is demanded.

[23]Fleiner et al. (2019) worked with a matching model and considered competitive equilibrium with personalized pricing, but their arguments also apply in exchange economies without personalized pricing. See also Schlegel (2022).

For quasilinear utility functions, since Marshallian demand is independent of the endowment and coincides with demand, gross substitutability is equivalent to (net) substitutability. To understand the difference between gross and net substitutability in the presence of income effects, we compare the conditions in a setting in which agents have unit demand for goods.

Example 2 (Example 1 continued)**.** Consider an agent, Martine, who owns a house i_1 and is considering selling it to purchase (at most) one of houses i_2 and i_3. If Martine experiences income effects, then her choice between i_2 and i_3 generally depends on the price she is able to procure for her house i_1. For example, if i_3 is a more luxurious house than i_2, then Martine may only demand i_3 if the value of her endowment is sufficiently large—i.e., if the price of her house i_1 is sufficiently high. As a result, when Martine is endowed with i_1, she does not generally have gross substitutes preferences: increases in the price of i_1 can lower Martine's demand for i_2. That is, Martine can regard i_2 as a gross complement to i_1. In contrast, Martine has net substitutes preferences—no compensated increase in the price of i_1 could make Martine stop demanding i_2—a condition that holds generally in the housing market economy.[24] And unlike net substitutability, gross substitutability generally depends on endowments: if Martine were not endowed a house, she would have gross substitutes preferences (Kaneko, 1982, 1983; Demange and Gale, 1985).

While Example 2 shows that net substitutability does not imply gross substitutability, it turns out that gross substitutability implies net substitutability.

Proposition 1. *If agent j demands at most one unit of each good, and there exists an endowment $\mathbf{w}_I \in \{0,1\}^I$ of goods such that U^j is a gross substitutes utility function at endowment $\mathbf{w}$ for all money endowments w_0, then U^j is a net substitutes utility function.*

Proposition 1 and Example 2 show that gross substitutability (at any one endowment of goods) implies net substitutability but places additional restrictions on income effects. In particular, the existence of competitive equilibrium under gross substitutability is a special case of Corollary 1.[25] But Corollary 1 is more general: as Example 2 shows, net substitutability allows for gross complementarities between goods that arise due to income effects, in addition to gross substitutability. As discussed in Section 3, the possibility of weakening gross substitutability to a condition on substitution effects alone that is sufficient for the existence of competitive equilibrium is an example of the more general point that substitution effects fundamentally determine whether equilibrium exists.

[24] Danilov et al. (2001, Example 2) also showed the connection between Quinzii's (1984) housing market economy and a substitutability condition, but formulated their discussion in terms of the shape of the convex hull at domains at which demand is multi-valued instead of net substitutability. Their discussion is equivalent to ours by Corollary 5 in Danilov, Koshevoy, and Lang (2003) and Remark 1.

[25] As Proposition 1 requires gross substitutability for all money endowments, the existence result of Fleiner et al. (2019) is not strictly a special case of Corollary 1. Moreover, Fleiner et al. (2019) also allowed for frictions such as transaction taxes and commissions in their existence result.

When agents see goods as gross substitutes, there are stronger results than existence: in this case, iteratively increasing the prices of over-demanded goods leads to a competitive equilibrium (Kelso and Crawford, 1982; Gul and Stacchetti, 2000; Milgrom, 2000; Fleiner et al., 2019). We next illustrate by example how the distinction between gross substitutability and net substitutability affects whether dynamic auctions can find equilibrium when agents can demand multiple goods. To do so, we develop a convenient functional form for net substitutes utility functions.

Example 3 (Quasilogarithmic Utility)**.** Given a function $V^j_{\mathrm{Q}} : X^j_I \to (-\infty, 0)$, which we call a *quasivaluation*,[26] and letting $\underline{u}^j = -\infty$, $\overline{u}^j = \infty$, and $\underline{x}^j_0 = 0$, there is a *quasilogarithmic* utility function given by

$$U^j(\mathbf{x}) = \log x_0 - \log(-V^j_{\mathrm{Q}}(\mathbf{x}_I)).$$

The Hicksian valuation at utility level u is $V^j_{\mathrm{H}}(\mathbf{x}_I; u) = e^u V^j_{\mathrm{Q}}(\mathbf{x}_I)$—a positive linear transformation of the quasivaluation V^j_{Q}. By Remark 1, it follows that a quasilogarithmic utility function U^j is a net substitutes utility function if and only if the quasivaluation V^j_{Q} is a substitutes valuation.

Our example features two agents with net substitutes quasilogarithmic utility functions, and a third agent who has a quasilinear utility function.

Example 4 (Gross Substitutability versus Net Substitutability with Multiple Goods, and Finding Competitive Equilibrium)**.** There are two goods and the total endowment of goods is $\mathbf{y}_I = (1, 1)$. There are three agents, which we call j, j', and k. Each agent's feasible set of consumption bundles of goods is $X^j_I = \{0, 1\}^2$.

Agents j and j' share the same utility function, which is quasilogarithmic with quasivaluation given by

$$V^j_{\mathrm{Q}}(\mathbf{x}_I) = V^{j'}_{\mathrm{Q}}(\mathbf{x}_I) = 6x_1 + 3x_2 - 10.$$

They have money endowment $w^j_0 = w^{j'}_0 = 10$ and goods endowment $\mathbf{w}^j_I = \mathbf{w}^{j'}_I = (0, 0)$. Figure 1 graphically depicts the demand of agents j and j', and the analysis of the example. Agent k has a quasilinear utility function with $V^j(\mathbf{x}_I) = 0$, and goods endowment $\mathbf{w}^k_I = (1, 1)$. Intuitively, agents j and j' are the buyers, and agent k the seller, in an auction of two indivisible goods.

Agent k clearly has a substitutes valuation. The quasivaluation of agents j and j' is a substitutes valuation as well; it follows that j and j' have net substitutes utility functions (Example 3). Hence, Corollary 1 guarantees that competitive equilibrium exists.

[26]Here, we call V^j_{Q} a quasivaluation, and denote it by V^j_{Q} instead of V^j, to distinguish it from the valuation of an agent with a quasilinear utility function.

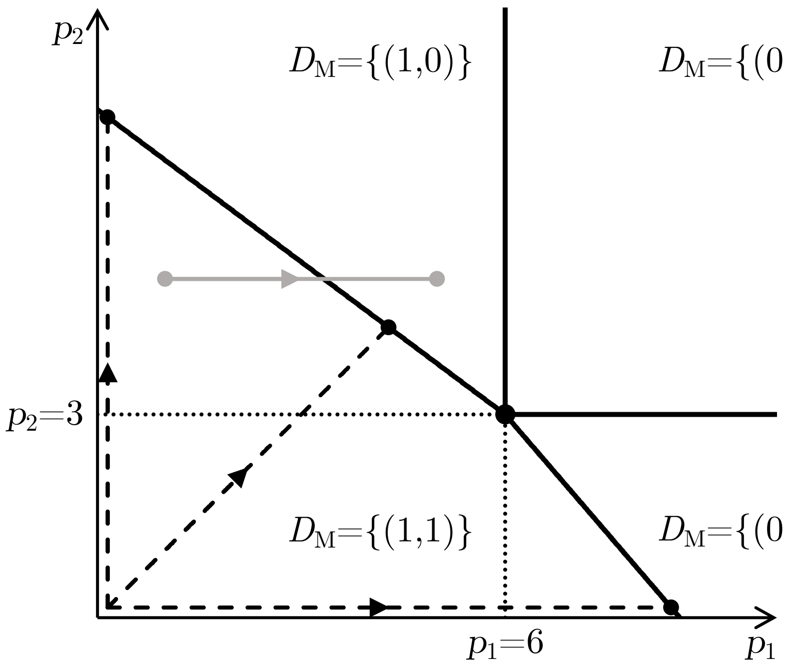

FIGURE 1. Demand, Competitive Equilibrium, and Dynamic Auctions in Example 4. The figure depicts the Marshallian demand of agents j and j' for endowment $\mathbf{w}_I^j = \mathbf{w}_I^{j'} = (0,0)$, which we denote by D_{M}; the solid black lines partition the space of prices based on which bundles are demanded. There is a unique price vector at which bundles $(1,0)$ and $(0,1)$ are both demanded, which is the competitive equilibrium price vector $\mathbf{p}_I = (6,3)$, as marked by the dotted lines. Starting from a price vector just above $(0,0)$, increasing the price of good 1 (resp. good 2) until demand for it falls—along the horizontal (resp. vertical) dashed line—overshoots the equilibrium price of good 1 (resp. good 2). Increasing the prices of both goods at the same rate until demand for one of them falls—along the diagonal dashed line—overshoots the equilibrium price of good 2 as well. These auctions fail as there are gross complementarities; for example, the increase in the price of good 1 drawn in gray changes Marshallian demand from $(1,1)$ to $(1,0)$—a decrease in Marshallian demand for good 2.

In fact, there is a unique competitive equilibrium price vector, namely $\mathbf{p}_I = (6,3)$. This price vector supports multiple competitive equilibria: in one, the allocation of goods is given by $\mathbf{x}_I^j = (1,0)$, $\mathbf{x}_I^{j'} = (0,1)$, and $\mathbf{x}_I^k = (0,0)$.

However, standard ascending auctions do not find competitive equilibrium. To understand why, suppose that the price vector starts at just above $(0,0)$. Then, both goods are over-demanded. If only the price of good 1 were increased, then it would reach $8\frac{4}{7}$ before demand for good 1 fell. But then the price of good 1 would have to decrease by $2\frac{4}{7}$ to reach a competitive equilibrium. Similarly, if only the price of good 2 were increased, then it would reach $7\frac{1}{2}$ before demand for good 1 fell—and then the price of good 2 would have to decrease by $4\frac{1}{2}$ to reach a competitive equilibrium. On the other hand, if the prices of both goods were increased at the same rate, the price vector would reach $(4\frac{2}{7}, 4\frac{2}{7})$ before demand for either good fell—and then the price of good 2 would have to decrease by $1\frac{2}{7}$ to reach a competitive equilibrium.[27]

One ascending auction that would find a competitive equilibrium would be one that increased the price of good 1 at twice the rate it increased the price of good 2. But this approach relies on *ex ante* knowledge of the competitive equilibrium price vector: it would fail, for example, if the values of the two goods were switched.

These issues with ascending auction arise because agents j and j' do not have gross substitutes utility functions. Indeed, as the price vector changes from $(1,5)$ to $(5,5)$, agent j's Marshallian demand changes from $(1,1)$ to $(1,0)$—so increasing the price of good 1 can decrease Marshallian demand for good 2.

Thus, when there are gross complementarities between goods, increases in the price of an over-demanded good can lead to other goods being under-demanded due to an income effect. So, even though competitive equilibrium is guaranteed to exist when agents see goods as net substitutes, it may not be possible to find a competitive equilibrium using a monotone, dynamic auction unless agents in fact see goods as gross substitutes. In particular, approaches to showing equilibrium existence using ascending auctions or tâtonnement (Crawford and Knoer, 1981; Kelso and Crawford, 1982; Gul and Stacchetti, 2000; Milgrom, 2000; Fleiner et al., 2019), which work under gross substitutability, cannot be used to prove Corollary 1. Thus, Example 4 illustrates that there is no fundamental connection between the existence of competitive equilibrium and tâtonnement—even when goods are substitutable.

4.4. **Net Substitutability as a Maximal Domain.** In general, net substitutability forms a maximal domain for the existence of competitive equilibrium. Specifically, if an agent does not have net substitutes preferences, then competitive equilibrium may not exist when the other agents have substitutes quasilinear preferences. Technically, we require that one unit of each good be present among agents' endowments (i.e., that $y_i = 1$ for all goods i) as complementarities between goods that are not present are irrelevant for the existence of competitive equilibrium.

Corollary 2. *Suppose that $y_i = 1$ for all goods i. If $|J| \geq 2$, agent j demands at most one unit of each good, and U^j is not a net substitutes utility function, then there exist substitutes valuations $V^k : \{0,1\}^I \to \mathbb{R}$ for agents $k \neq j$, and a feasible endowment profile for which no competitive equilibrium exists.*

[27]Jagadeesan and Teytelboym (2021) have provided an example in which descending auctions also fail to find competitive equilibrium even under net substitutability.

Corollary 2 entails that any domain of preferences that contains all substitutes quasilinear preferences and guarantees the existence of competitive equilibrium must lie within the domain of net substitutes preferences. Therefore, Corollaries 1 and 2 suggest that net substitutability is the most general way to incorporate income effects into a substitutability condition that ensures the existence of competitive equilibrium for all endowments.

By contrast, the relationship between the nonexistence of competitive equilibrium and failures of gross substitutability depends on why gross substitutability fails. Gross substitutability can fail due to substitution effects that reflect net complementarities or due to income effects. If the failure of gross substitutability reflects a net complementarity, then Corollary 2 tells us that competitive equilibrium may not exist if the other agents have substitutes quasilinear preferences. On the other hand, if the failure of gross substitutability is only due to income effects then, as in Example 4, Corollary 1 tells us that competitive equilibrium exists if the other agents have net substitutes preferences (e.g., substitutes quasilinear preferences).

Corollary 2 is an immediate consequence of the Equilibrium Existence Duality and the fact that substitutability defines a maximal domain for the existence of competitive equilibrium with transferable utility.

Fact 3. *Suppose that $y_i = 1$ for all goods i. If $|J| \geq 2$, agent j demands at most one unit of each good, and V^j is not a substitutes valuation, then there exist substitutes valuations $V^k : \{0,1\}^I \to \mathbb{R}$ for agents $k \neq j$ for which no competitive equilibrium exists.*[28]

Proof of Corollary 2. By Remark 1, there exists a utility level u at which agent j's Hicksian valuation $V^j_{\mathrm{H}}(\cdot; u)$ is not a substitutes valuation. Fact 3 implies that there exist substitutes valuations $V^k : \{0,1\}^I \to \mathbb{R}$ for agents $k \neq j$, for no competitive equilibrium would exist with transferable utility if agent j's valuation were $V^j_{\mathrm{H}}(\cdot; u)$. With those valuations V^k for agents $k \neq j$, a feasible endowment profile clearly exists, and hence the "only if" direction of Theorem 1 implies that there exists a feasible endowment profile for which no competitive equilibrium exists. □

4.5. **Relationship to Danilov et al. (2001).** Corollary 1 is related to a result of Danilov et al. (2001) that was formulated in different terms. Assuming that $\mathbf{0}_I \in X^j_I$ and that $\underline{x}^j_0 = 0$ for all agents j, Danilov et al. (2001) considered the functions

$$q^j_m(\mathbf{x}_I) = U^j(\cdot, \mathbf{x}_I)^{-1}\left(U^j(m, \mathbf{0}_I)\right)$$

[28]Fact 3 is a version of Theorem 2 in Gul and Stacchetti (1999) and Theorem 4 in Yang (2017) that applies when X^j_I can be strictly contained in $\{0,1\}^I$, as well as a version of Theorem 7 in Hatfield et al. (2013) for exchange economies. For sake of completeness, we give a proof of Fact 3 in Appendix D.

and introduced the domains of preferences such that $q^j_m(\mathbf{x}_I)$ is "$\mathscr{D}$-convex" for all $m > 0$, in that

$$\left(\operatorname*{arg\,min}_{\mathbf{x}_I \in X^j_I}\{\mathbf{p}_I \cdot \mathbf{x}_I + q^j_m(\mathbf{x}_I)\}\right) \in \mathscr{D}$$

for all price vectors $\mathbf{p}_I$, where $\mathscr{D}$ is a "class of discrete convexity." Thus, in our terminology, Danilov et al. (2001) considered the domains of preferences such that Hicksian demand sets are elements of particular families $\mathscr{D}$ of subsets of $\mathbb{Z}^I$. Their Theorem 2 shows that, under some technical conditions, a sufficient (but not a necessary) condition for equilibrium existence is that all agents' preferences are in such a domain.

While our Equilibrium Existence Duality leads to existence results beyond domains related to classes of discrete convexity, Theorem 2 in Danilov et al. (2001) is connected to our Corollary 1. To understand the connection, note that as discussed in Section 5 in Danilov et al. (2001), an important class of discrete convexity is the class $\mathscr{IGP}$ of "integral generalized polymatroids" (in the sense of Frank (1984)). Danilov, Koshevoy, and Lang (2003) and Fujishige and Yang (2003) showed that if an agent has a quasilinear utility function and demands at most one unit of each good, then her demand sets are all integral generalized polymatroids if and only if her valuation is a substitutes valuation. Applying Danilov, Koshevoy, and Lang's (2003) and Fujishige and Yang's (2003) results to the Hicksian valuations $V^j_{\mathrm{H}}(\cdot; U^j(m, \mathbf{0}_I)) = -q^j_m(\cdot)$, we see that if agent j demands at most one unit of each good is demanded, then $q^j_m(\cdot)$ is $\mathscr{IGP}$-convex for all $m > 0$ if and only if each of j's Hicksian valuation is a substitutes valuation—which happens if and only if U^j is a net substitutes utility function (by Remark 1). However, our proof of Corollary 1 via the Equilibrium Existence Duality highlights the role of substitution effects in determining whether equilibrium exists.[29]

Importantly, the Equilibrium Existence Duality also leads to additional results that are beyond the scope of Danilov et al.'s (2001) methods. Indeed, while Danilov et al.'s (2001) methods provide existence results only for classes of discrete convexity, the Equilibrium Existence Duality allows us to transport *any necessary or sufficient condition* for the existence of equilibrium in transferable utility economies over to settings with income effects. For example, the Equilibrium Existence Duality can be applied to derive maximal domain results, such as Corollary 2. The next section gives further applications of the Equilibrium Existence

[29]Moreover, Corollary 1 does not follow from Theorem 2 in Danilov et al. (2001) due to differences in technical assumptions. Indeed, Danilov et al. (2001) allowed for production of goods from money but required that all nonnegative consumption bundles be feasible for consumers. These assumptions together rule out the technological constraints on production that have featured prominently in Hatfield et al. (2013) and Fleiner et al. (2019).

Duality to both sufficient conditions and a necessary condition that go beyond the classes of preferences that Danilov et al.'s (2001) methods can cover.[30]

5. Beyond Substitutes: Further Applications of the Equilibrium Existence Duality

While Section 4 focused on the case of substitutes, the Equilibrium Existence Duality can be used to extend *any* necessary or sufficient condition for the existence of competitive equilibrium from transferable utility economies to settings with income effects. In this section, we provide two further applications of the Equilibrium Existence Duality that incorporate net complementarities. Given the fundamental role of substitution effects in determining equilibrium existence, these applications lead to new conditions on substitution effects (or, equivalently, Hicksian valuations) that are necessary or sufficient for the existence of competitive equilibrium. As discussed in Section 4.5, these applications go beyond the classes of preferences that Danilov et al.'s (2001) methods can cover.

5.1. **A Necessary and Sufficient Condition.** The Equilibrium Existence Duality leads to an extension of Bikhchandani and Mamer's (1997) well-known necessary and sufficient condition for the existence of competitive equilibrium exchange economies to settings with income effects. We first recall Bikhchandani and Mamer's (1997) result.

Fact 4 (Bikhchandani and Mamer, 1997)**.** *Suppose that utility is transferable and that a feasible endowment profile exists. Competitive equilibrium exists if and only if the linear program*

$$\begin{aligned} &\max_{\left(\alpha^j \in \mathbb{R}_{\geq 0}^{X_I^j}\right)_{j \in J}} \sum_{j \in J} \sum_{\mathbf{x}_I \in X_I^j} \alpha_{\mathbf{x}_I}^j V^j(\mathbf{x}_I) \\ &\textit{subject to} \quad \sum_{\mathbf{x}_I \in X_I^j} \alpha_{\mathbf{x}_I}^j = 1 \textit{ for all } j \in J \quad \textit{and} \quad \sum_{j \in J} \sum_{\mathbf{x}_I \in X_I^j} \alpha_{\mathbf{x}_I}^j \mathbf{x}_I = \mathbf{y}_I \end{aligned} \tag{4}$$

has an integer maximizer.

[30]On the other hand, one can apply the Equilibrium Existence Duality to deduce a version of Danilov et al.'s (2001) results from Baldwin and Klemperer's (2019) "Unimodularity Theorem" and illuminate them in terms of Hicksian demand; see Baldwin, Edhan, Jagadeesan, Klemperer, and Teytelboym (2021) for this application. To understand the connection to Baldwin and Klemperer (2019), note that Theorem 4 in Danilov et al. (2001) shows that the classes of discrete convexity relevant to the equilibrium existence problem are generated by sets $\mathscr{R}$ of integer vectors that are *unimodular* in the sense that every square matrix whose column vectors lie in $\mathscr{R}$ has determinant 0 or ± 1 (plus an extra condition if $\mathscr{R}$ does not span $\mathbb{R}^I$). The quasilinear utility functions in the corresponding classes of preferences correspond to Baldwin and Klemperer's (2019) "unimodular demand types" (by Proposition 2.20 in Baldwin and Klemperer (2019)), which are the domains for which their "Unimodularity Theorem" demonstrates equilibrium existence.

Fact 4 shows that with transferable utility, the equilibrium existence problem is closely related to a welfare-maximization linear programming problem. By the Equilibrium Existence Duality, Fact 4 immediately extends to give a necessary and sufficient condition for competitive equilibria to exist for all endowment allocations.

Corollary 3. *Suppose that a feasible endowment profile exists. Competitive equilibria exist for all feasible endowment profiles if and only if, for each profile* $(u^j)_{j \in J}$ *of utility levels, the Bikhchandani–Mamer linear program (4) has an integer maximizer when each agent* j*'s valuation is taken to be her Hicksian valuation* $V^j_{\mathrm{H}}(\cdot; u^j)$*.*

Corollary 3 shows that like the case of transferable utility, existence is closely related to linear programming. However, unlike the transferable utility case, existence depends on a *family* of linear programming problems.

As Corollary 3 applies for fixed preferences and total endowment of goods, it leads to new conditions for the existence of competitive equilibrium. For example, in terms of sufficient conditions, Bikhchandani and Mamer's (1997) result has been applied by Candogan et al. (2015) to demonstrate existence for a class of valuations that allows for substitutability and complementarity. Candogan et al.'s (2015) result assumes that the total endowment of goods consists of one unit of each good ($y_i = 1$ for all goods i). In this case, Candogan et al. (2015) showed that competitive equilibrium in transferable utilities economies in which each agent's valuation is a *sign-consistent tree valuation*. By Theorem 1, it follows that even with income effects, competitive equilibrium exists if all agents' Hicksian valuations at all utility levels are sign-consistent tree valuations.[31]

Corollary 3 also readily leads to necessary conditions for the existence of competitive equilibrium.

5.2. **Matching with Complementarities.** The Equilibrium Existence Duality can also be used to deduce a result on matching markets with net complementarities from results of Rostek and Yoder (2020b). In a matching model with transferable utility, Rostek and Yoder (2020b) showed that competitive equilibria always exist when agents view the primitive contracts by which they interact as (gross) complements. Here, a "primitive contract" specifies all aspects of an interaction among a group of agents except for the payments between

[31]This consequence of Corollary 3 goes beyond the classes of preferences for which Danilov et al.'s (2001) methods are applicable. Indeed, Section 3.3 of the Supplemental Appendix of Candogan et al. (2015) shows that their existence result does not follow from Baldwin and Klemperer's (2019) existence result, which covers precisely the classes of preferences for which Danilov et al.'s (2001) methods are applicable in transferable utility economies (see Footnote 30). Intuitively, Section 3.3 of the Supplemental Appendix of Candogan et al. (2015) shows that existence is guaranteed when agents' valuations are sign-consistent tree valuations only when the total endowment of goods $\mathbf{y}_I$ consists of at most 1 unit of each good; by contrast, Danilov et al.'s (2001) methods always demonstrate existence for all total endowments of goods $\mathbf{y}_I$ (for which endowment allocations exist) when they apply.

members of the group. Rostek and Yoder (2020a) showed that when contracts do not induce externalities on non-participants, Rostek and Yoder's (2020b) transferable utility matching model can be embedded in an exchange economy. As a result, it follows from Theorem 1 that competitive equilibria always exist in a similar model with income effects as long as agents view primitive contracts as net complements.[32]

6. Conclusion

Our "Hicksian economies" are useful tools for analyzing economies with indivisible goods; in particular, they isolate substitution effects. They are based on representing Hicksian demand in terms of quasilinear maximization problems for "Hicksian valuations." The Equilibrium Existence Duality shows that competitive equilibrium exists (for all endowment allocations) if and only if competitive equilibrium exists in each Hicksian economy. An application is that it is net substitutability, not gross substitutability, that is relevant to the existence of equilibrium. Further applications give new existence results beyond the case of substitutes. In short, with income effects, just as without them, existence does not depend on agents seeing goods as substitutes; rather, substitution effects determine whether equilibrium exists.

Our work also has implications for auction design. First, our perspective of analyzing preferences by using Hicksian valuations may yield new approaches for extending auction bidding languages to allow for income effects. Second, our equilibrium existence results suggest that some sealed-bid auctions with competitive equilibrium pricing may work well for indivisible goods even in the presence of financing constraints. One set of examples are Product-Mix Auctions, such as the one implemented by the Bank of England[33]—these implement competitive equilibrium allocations assuming that the submitted sealed bids represent bidders' actual preferences, since truth-telling is a reasonable approximation in these auctions when there are sufficiently many bidders. As monotone, dynamic auctions may not find competitive equilibrium in the presence of income effects, the approach of finding competitive equilibrium based on a single round of sealed bids seems especially useful.

[32]This application of the Equilibrium Existence Duality is also beyond the scope of Danilov et al.'s (2001) methods. Indeed, Example 1 in Rostek and Yoder (2020a) shows that Rostek and Yoder's (2020b) existence result does not follow from Baldwin and Klemperer's (2019) existence result, which covers precisely the classes of preferences for which Danilov et al.'s (2001) methods are applicable in transferable utility economies (see Footnote 30).

[33]See Klemperer (2010, 2018) and Baldwin and Klemperer (in preparation). IMF staff recently proposed a Product-Mix Auction for bidders with budget constraints (for sovereign debt restructuring—see Willems (2021)), and Iceland planned a similar auction with budget constraints (for buying up "offshore" funds—see Klemperer (2018)) but those auctions were for divisible goods.

Appendix A. Proof of Theorem 1 and Lemma 2

We prove the following result, which combines Theorem 1 and Lemma 2.

Theorem A.1. *If a feasible endowment profile exists, then the following are equivalent.*

(I) Competitive equilibria exist for all feasible endowment profiles.

(II) For each Pareto-efficient allocation $(\mathbf{x}^j)_{j \in J}$ with $\sum_{j \in J} \mathbf{x}_I^j = \mathbf{y}_I$, there exists a price vector $\mathbf{p}_I$ such that $\mathbf{x}^j \in D_{\mathrm{M}}^j(\mathbf{p}_I, \mathbf{x}^j)$ for all agents j.

(III) Competitive equilibria exist in the Hicksian economies for all profiles of utility levels.

The remainder of this appendix is devoted to the proof of Theorem A.1. The proof uses the following simple lemma in several places.

Lemma A.2. *For all agents j and bundles $\mathbf{x}_I^j \in X_I^j$, the function $V_{\mathrm{H}}^j(\mathbf{x}_I; \cdot)$ is continuous and strictly decreasing.*[34]

Proof. Follows from the Inverse Function Theorem because $U^j(\cdot, \mathbf{x}_I)$ is continuous, strictly increasing, and satisfies Condition (1). $\square$

A.1. **Proof of the (I) $\implies$ (II) Implication in Theorem A.1.** The proof of this implication is essentially identical to the proof of Theorem 3 in Maskin and Roberts (2008). Consider a Pareto-efficient allocation $(\mathbf{x}^j)_{j \in J}$ with $\sum_{j \in J} \mathbf{x}^j = \mathbf{y}_I$.

Let agent j's endowment be $\mathbf{w}^j = \mathbf{x}^j$. By Statement (I) in the theorem, there exists a competitive equilibrium—say consisting of the price vector $\mathbf{p}_I$ and the allocation $(\hat{\mathbf{x}}_I^j)_{j \in J}$ of goods. By the definition of competitive equilibrium, we have that $\hat{\mathbf{x}}_I^j \in D_{\mathrm{M}}^j\left(\mathbf{p}_I, \mathbf{x}_I^j\right)$ for all agents j. In particular, letting $\hat{x}_0^j = x_0^j - \mathbf{p}_I \cdot (\hat{\mathbf{x}}_I^j - \mathbf{x}_I^j)$ for each agent j, we have that $\sum_{j \in J} \hat{\mathbf{x}}^j = \sum_{j \in J} \mathbf{x}^j$ and that $U^j(\hat{\mathbf{x}}^j) \geq U^j(\mathbf{x}^j)$ for all agents j. As the allocation $(\mathbf{x}^j)_{j \in J}$ is Pareto-efficient, we must have that $U^j(\hat{\mathbf{x}}^j) = U^j(\mathbf{x}^j)$ for all agents j. It follows that $\mathbf{x}_I^j \in D_{\mathrm{M}}^j(\mathbf{p}_I, \mathbf{x}^j)$ for all agents j—as desired.

A.2. **Proof of the (II) $\implies$ (III) Implication in Theorem A.1.** The key to the proof is the following characterization of Pareto-efficiency in terms of the Hicksian valuations.

Claim A.3. *Let $(u^j)_{j \in J}$ be a profile of utility levels. If an allocation $(\mathbf{x}_I^j)_{j \in J} \in \bigtimes_{j \in J} X_I^j$ of goods with $\sum_{j \in J} \mathbf{x}_I^j = \mathbf{y}_I$ maximizes*

$$\sum_{j \in J} V_{\mathrm{H}}^j\left(\mathbf{x}_I^j; u^j\right)$$

over all allocations $(\hat{\mathbf{x}}_I^j)_{j \in J} \in \bigtimes_{j \in J} X_I^j$ of goods with $\sum_{j \in J} \hat{\mathbf{x}}_I^j = \mathbf{y}_I$, then, writing $x_0^j = -V_{\mathrm{H}}^j\left(\mathbf{x}_I^j; u^j\right)$ for each agent j, the allocation $(\mathbf{x}^j)_{j \in J}$ is Pareto-efficient.[35]

[34]Lemma A.2 is a version of Proposition 6 in Luenberger (1992a) for settings with indivisible goods and a fixed numéraire.

[35]Claim A.3 is a version of Lemma 3.2 in Luenberger (1992b) for settings with indivisible goods.

Proof. Note that $U^j(\mathbf{x}^j) = u^j$ for all agents j by construction. Consider any allocation $(\hat{\mathbf{x}}^j)_{j\in J} \in \bigtimes_{j\in J} X^j$ with $\sum_{j\in J} \hat{\mathbf{x}}_I^j = \mathbf{y}_I$, and $U^j(\hat{\mathbf{x}}^j) \geq U^j(\mathbf{x}^j) = u^j$ for all agents j with strict inequality for some $j = j_1$. As $V_{\mathrm{H}}^j\left(\hat{\mathbf{x}}_I^j;\cdot\right)$ is strictly decreasing for each agent j (by Lemma A.2), we must have that

$$\hat{x}_0^j = -V_{\mathrm{H}}^j\left(\hat{\mathbf{x}}_I^j; U^j\left(\hat{\mathbf{x}}^j\right)\right) \geq -V_{\mathrm{H}}^j\left(\hat{\mathbf{x}}_I^j; u^j\right)$$

for all agents j with strict inequality for $j = j_1$. Hence, we must have that

$$\sum_{j\in J} \hat{x}_0^j > -\sum_{j\in J} V_{\mathrm{H}}^j\left(\hat{\mathbf{x}}_I^j; u^j\right) \geq -\sum_{j\in J} V_{\mathrm{H}}^j\left(\mathbf{x}_I^j; u^j\right) = \sum_{j\in J} x_0^j,$$

where the second inequality follows from the definition of $(\mathbf{x}_I^j)_{j\in J}$, so the allocation $(\mathbf{x}^j)_{j\in J}$ cannot be Pareto-dominated. □

To complete the proof, let $(u^j)_{j\in J}$ be a profile of utility levels. Let $(\mathbf{x}_I^j)_{j\in J} \in \bigtimes_{j\in J} X_I^j$ be as in the statement of Claim A.3; such an allocation exists because each set X_I^j is finite and a feasible endowment profile exists. Letting $x_0^j = -V_{\mathrm{H}}^j\left(\mathbf{x}_I^j; u^j\right)$ for each agent j, by Claim A.3 and Statement (II) in the theorem, there exists a price vector $\mathbf{p}_I$ such that $\mathbf{x}_I^j \in D_{\mathrm{M}}^j(\mathbf{p}_I, \mathbf{x}^j)$ for all agents j. Fact 1 implies that $\mathbf{x}_I^j \in D_{\mathrm{H}}^j(\mathbf{p}_I; u^j)$ for all agents j. By Lemma 1, it follows that the price vector $\mathbf{p}_I$ and the allocation $\left(\mathbf{x}_I^j\right)_{j\in J}$ of goods comprise a competitive equilibrium in the Hicksian economy for the profile $(u^j)_{j\in J}$ of utility levels.

A.3. **Proof of the (III) $\Longrightarrow$ (I) Implication in Theorem A.1.** Let $(\mathbf{w}^j)_{j\in J}$ be an endowment allocation. Given a profile $\mathbf{u} = (u^j)_{j\in J}$ of utility levels, let

$$T(\mathbf{u}) = \left\{ \left(\begin{array}{c} \mathbf{p}_I \cdot (\mathbf{x}_I^j - \mathbf{w}_I^j) \\ -V_{\mathrm{H}}^j\left(\mathbf{x}_I^j; u^j\right) - w_0^j \end{array} \right)_{j\in J} \;\middle|\; \begin{array}{l} \left(\mathbf{p}_I, (\mathbf{x}_I^j)_{j\in J}\right) \text{ is a competitive} \\ \text{equilibrium in the Hicksian economy} \\ \text{for the profile } (u^j)_{j\in J} \text{ of utility levels} \end{array} \right\}$$

denote the set of profiles of net expenditures over all competitive equilibria in the Hicksian economy for the profile $(u^j)_{j\in J}$ of utility levels.

Claim A.4. *Under Statement (III) in Theorem A.1, there exists a profile* $\mathbf{u} = (u^j)_{j\in J}$ *of utility levels such that* $\mathbf{0} \in T(\mathbf{u})$.

To complete the proof of the (III) $\Longrightarrow$ (I) implication in Theorem A.1 from Claim A.4, note that Claim A.4 implies that there exists a profile $\mathbf{u} = (u^j)_{j\in J}$ of utility levels and a competitive equilibrium $(\mathbf{p}_I, (\mathbf{x}_I^j)_{j\in j})$ in the corresponding Hicksian economy with

$$\mathbf{p}_I \cdot (\mathbf{x}_I^j - \mathbf{w}_I^j) - V_{\mathrm{H}}^j\left(\mathbf{x}_I^j; u^j\right) = w_0^j \tag{A.1}$$

for all agents j. Lemma 1 implies that $\mathbf{x}_I^j \in D_{\mathrm{H}}^j(\mathbf{p}_I; u^j)$ for all agents j, and we have that

$$U^j\left(w_0^j - \mathbf{p}_I \cdot (\mathbf{x}_I^j - \mathbf{w}_I^j), \mathbf{x}_I^j\right) = U^j\left(-V_{\mathrm{H}}^j\left(\mathbf{x}_I^j; u^j\right), \mathbf{x}_I^j\right) = u^j$$

for all agents j by Equation (A.1) and the definition of V^j_{H}. It follows from Fact 1 that $\mathbf{x}^j_I \in D^j_{\mathrm{M}}(\mathbf{p}_I, \mathbf{w}^j)$ for all agents j, so the price vector $\mathbf{p}_I$ and the allocation $(\mathbf{x}^j_I)_{j\in J}$ of goods comprise a competitive equilibrium in the original economy for the endowment allocation $(\mathbf{w}^j)_{j\in J}$.

The remainder of this appendix is devoted to the proof of Claim A.4. We first restrict the domain of T to a product of compact intervals. Intuitively, this restriction corresponds to the fact that in equilibrium, agents cannot obtain lower utility than the utility of their endowment, nor higher utility than the highest utility level they could get if they were paid all the social surplus in the economy. We show that the correspondence T is upper hemicontinuous and has nonempty compact, convex values on the restricted domain, and then apply a topological fixed-point argument to prove the claim.

Formally, for each agent j, we define a utility level $u^j_{\min} = U^j(\mathbf{w}^j)$, which is defined due to the feasibility of $(\mathbf{w}^j)_{j\in J}$, and let

$$K^j = w^j_0 + \max_{\mathbf{x}_I \in X^j_I} V^j_{\mathrm{H}}\left(\mathbf{x}_I; u^j_{\min}\right) \geq w^j_0 + V^j_{\mathrm{H}}\left(\mathbf{w}^j_I; u^j_{\min}\right) = 0$$

Furthermore, let $K = 1 + \sum_{j\in J} K^j$ and let

$$u^j_{\max} = \max_{\mathbf{x}_I \in X^j_I} U^j\left(w^j_0 + K, \mathbf{x}_I\right).$$

We begin by proving that the correspondence $T : \bigtimes_{j\in J}[u^j_{\min}, u^j_{\max}] \rightrightarrows \mathbb{R}^J$ is upper hemicontinuous and has compact, convex values. We actually give explicit bounds for the range of T. Let

$$\overline{M} = -\min_{j\in J}\left\{V^j_{\mathrm{H}}\left(\mathbf{w}^j_I; u^j_{\max}\right) + w^j_0\right\}$$

and let

$$\underline{M} = -(|J|-1)\overline{M} - \sum_{j\in J}\left(w^j_0 + \max_{\mathbf{x}_I\in X^j_I}\left\{V^j_{\mathrm{H}}\left(\mathbf{x}_I; u^j_{\min}\right)\right\}\right).$$

Claim A.5. *The correspondence $T : \bigtimes_{j\in J}[u^j_{\min}, u^j_{\max}] \rightrightarrows \mathbb{R}^J$ is upper hemicontinuous and has compact, convex values and range contained in $[\underline{M}, \overline{M}]^J$.*

The proof of Claim A.5 uses the following technical description of T.

Claim A.6. *Let $\mathbf{u} = (u^j)_{j\in J} \in \bigtimes_{j\in J}[u^j_{\min}, u^j_{\max}]$ be a profile of utility levels and let $(\mathbf{x}^j_I)_{j\in J} \in \bigtimes_{j\in J} X^j_I$ be an allocation of goods with $\sum_{j\in J}\mathbf{x}^j_I = \mathbf{y}_I$. If $(\mathbf{x}^j_I)_{j\in J}$ maximizes*

$$\sum_{j\in J} V^j_{\mathrm{H}}\left(\hat{\mathbf{x}}^j_I; u^j\right)$$

over all allocations $(\hat{\mathbf{x}}^j_I)_{j\in J} \in \bigtimes_{j\in J} X^j_I$ of goods with $\sum_{j\in J}\hat{\mathbf{x}}^j_I = \mathbf{y}_I$, then we have that

$$T(\mathbf{u}) = \left\{\left(\mathbf{p}_I\cdot(\mathbf{x}^j_I - \mathbf{w}^j_I) - V^j_{\mathrm{H}}\left(\mathbf{x}^j_I; u^j\right) - w^j_0\right)_{j\in J}\,\middle|\, \mathbf{p}_I \in \mathcal{P}\right\},$$

where

$$\mathcal{P} = \left\{\mathbf{p}_I \;\middle|\; \mathbf{p}_I \cdot \mathbf{x}_I^j - V_{\mathrm{H}}^j\left(\mathbf{x}_I^j; u^j\right) \leq \mathbf{p}_I \cdot \mathbf{x}'_I - V_{\mathrm{H}}^j\left(\mathbf{x}'_I; u^j\right) \text{ for all } j \in J \text{ and } \mathbf{x}'_I \in X_I^j\right\}.$$

Proof. By construction, we have that

$$\mathcal{P} = \left\{\mathbf{p}_I \;\middle|\; \begin{array}{c} (\mathbf{p}_I, (\mathbf{x}^j)_{j\in J}) \text{ is a competitive equilibrium in the} \\ \text{Hicksian economy for the profile } (u^j)_{j\in J} \text{ of utility levels} \end{array}\right\}.$$

A standard lemma regarding competitive equilibria in transferable utility economies shows that in the Hicksian economy for the profile $(u^j)_{j\in J}$ of utility levels, if $(\mathbf{p}_I, (\hat{\mathbf{x}}^j)_{j\in J})$ is a competitive equilibrium, then so is $\left(\mathbf{p}_I, (\mathbf{x}_I^j)_{j\in J}\right)$.[36] In this case, we have that

$$\mathbf{p}_I \cdot \mathbf{x}_I^j - V_{\mathrm{H}}^j\left(\mathbf{x}_I^j; u^j\right) = \mathbf{p}_I \cdot \hat{\mathbf{x}}_I^j - V_{\mathrm{H}}^j\left(\hat{\mathbf{x}}_I^j; u^j\right)$$

for all agents j. The claim follows. □

Proof of Claim A.5. It suffices to show that T has convex values, range contained in $[\underline{M}, \overline{M}]^J$, and a closed graph.

We first show that $T(\mathbf{u})$ is convex for all $\mathbf{u} \in \bigtimes_{j\in J}[u_{\min}^j, u_{\max}^j]$. We use the notation of Claim A.6 to prove this assertion. Note that $\mathcal{P}$ is the set of solutions to a set of linear inequalities, and is hence convex. Claim A.6 implies that $T(\mathbf{u})$ is the set of values of a linear function on $\mathcal{P}$—so it follows that $T(\mathbf{u})$ is convex as well.

We next show that $T(\mathbf{u}) \subseteq [\underline{M}, \overline{M}]^J$ holds for all $\mathbf{u} \in \bigtimes_{j\in J}[u_{\min}^j, u_{\max}^j]$. We continue to use the notation of Claim A.6. Let $\mathbf{u} \in \bigtimes_{j\in J}[u_{\min}^j, u_{\max}^j]$ and $\mathbf{t} \in T(\mathbf{u})$ be arbitrary. By Claim A.6, there exists $\mathbf{p}_I \in \mathcal{P}$ such that

$$t^j = \mathbf{p}_I \cdot (\mathbf{x}_I^j - \mathbf{w}_I^j) - V_{\mathrm{H}}^j\left(\mathbf{x}_I^j; u^j\right) - w_0^j$$

for all agents j. Note that for all agents j, we must have that

$$t^j \leq -V_{\mathrm{H}}^j\left(\mathbf{w}_I^j; u^j\right) - w_0^j \leq -V_{\mathrm{H}}^j\left(\mathbf{w}_I^j; u_{\max}^j\right) - w_0^j \leq \overline{M},$$

where the first inequality holds due to the definition of $\mathcal{P}$, the second inequality holds because $V_{\mathrm{H}}^j\left(\mathbf{w}_I^j; \cdot\right)$ is decreasing (by Lemma A.2), and the third inequality holds due to the definition of $\overline{M}$. Furthermore, as $\sum_{j\in J}\mathbf{x}_I^j = \mathbf{y}_I = \sum_{j\in J}\mathbf{w}_I^j$, we have that

$$\sum_{j\in J} t^j = -\sum_{j\in J}(V_{\mathrm{H}}^j\left(\mathbf{x}_I^j; u^j\right) + w_0^j).$$

[36]The lemma is due to Shapley (1964, page 3); see also Bikhchandani and Mamer (1997) and Hatfield et al. (2013). Jagadeesan et al. (2020, Lemma 1) proved the lemma in a setting in which agents can demand multiple units of each good and can have non-monotone valuations.

It follows that

$$\begin{aligned} t^j &= -\sum_{k\in J}(V_{\mathrm{H}}^k\left(\mathbf{x}_I^k;u^k\right)+w_0^k) - \sum_{k\in J\smallsetminus\{j\}} t^k \\ &\geq -\sum_{k\in J}(V_{\mathrm{H}}^k\left(\mathbf{x}_I^k;u_{\min}^k\right)+w_0^k) - \sum_{k\in J\smallsetminus\{j\}} t^k \\ &\geq -\sum_{k\in J}(V_{\mathrm{H}}^k\left(\mathbf{x}_I^k;u_{\min}^k\right)+w_0^k) - (|J|-1)\overline{M} \\ &\geq \underline{M} \end{aligned}$$

for all agents j, where the first inequality holds because $V_{\mathrm{H}}^k\left(\mathbf{x}_I^k;\cdot\right)$ is decreasing for each agent k (by Lemma A.2), the second inequality holds because $t^k \leq \overline{M}$ for all agents k, and the third inequality holds due to the definition of $\underline{M}$.

Last, we show that T has a closed graph. Our argument uses the following version of Farkas's Lemma.

Fact A.7 (Page 200 of Rockafellar, 1970[37])**.** *Let L_1, L_2 be disjoint, finite sets and, for each $\ell \in L_1 \cup L_2$, let $\mathbf{v}_I^\ell \in \mathbb{R}^I$ be a vector and let α_ℓ be a scalar. There exist scalars λ_ℓ for $\ell \in L_1 \cup L_2$ with $\lambda_\ell \geq 0$ for $\ell \in L_2$ such that*

$$\sum_{\ell\in L_1\cup L_2}\lambda_\ell \mathbf{v}_I^\ell = \mathbf{0} \quad \textit{and} \sum_{\ell\in L_1\cup L_2}\lambda_\ell\alpha_\ell < 0$$

if and only if there does not exist a vector $\mathbf{p}_I \in \mathbb{R}^I$ such $\mathbf{v}_I^\ell \cdot \mathbf{p}_I \leq \alpha_\ell$ for all $\ell \in L_1 \cup L_2$ with equality for all $\ell \in L_1$.

Consider a sequence $\mathbf{u}_{(1)}, \mathbf{u}_{(2)}, \ldots \in \bigtimes_{j\in J}[u_{\min}^j, u_{\max}^j]$ of profiles of utility levels. For each m, let $\mathbf{t}_{(m)} \in T(\mathbf{u}_{(m)})$. Suppose that $\mathbf{u}_{(m)} \to \mathbf{u}$ and $\mathbf{t}_{(m)} \to \mathbf{t}$ as $m \to \infty$. We need to show that $\mathbf{t} \in T(\mathbf{u})$.

As each set X_I^j is finite and a feasible endowment profile exists, by passing to a subsequence, we can assume that there exists an allocation $(\mathbf{x}_I^j)_{j\in J} \in \bigtimes_{j\in J} X_I^j$ of goods with $\sum_{j\in J}\mathbf{x}_I^j = \mathbf{y}_I$ that, for each m, maximizes

$$\sum_{j\in J} V_{\mathrm{H}}^j\left(\hat{\mathbf{x}}_I^j; u_{(m)}^j\right)$$

over all allocations $(\hat{\mathbf{x}}_I^j)_{j\in J} \in \bigtimes_{j\in J} X_I^j$ of goods with $\sum_{j\in J}\hat{\mathbf{x}}_I^j = \mathbf{y}_I$. By the continuity of $V_{\mathrm{H}}^j\left(\hat{\mathbf{x}}_I^j;u\right)$ in u for each agent j (Lemma A.2), the allocation $(\mathbf{x}_I^j)_{j\in J}$ of goods maximizes

$$\sum_{j\in J} V_{\mathrm{H}}^j\left(\hat{\mathbf{x}}_I^j; u^j\right)$$

over all allocations $(\hat{\mathbf{x}}_I^j)_{j\in J} \in \bigtimes_{j\in J} X_I^j$ of goods with $\sum_{j\in J}\hat{\mathbf{x}}_I^j = \mathbf{y}_I$.

[37]Theorem 22.1 in Rockafellar (1970) states the case of Fact A.7 in which $L_1 = \varnothing$. The version of Fact A.7 for $L_1 \neq \varnothing$ is left as an exercise on page 200 of Rockafellar (1970).

Suppose for sake of deriving a contradiction that $\mathbf{t} \notin T(\mathbf{u})$. Let $L_1 = J$ and let $L_2 = \bigcup_{j \in J} \{j\} \times X_I^j$. Define vectors $\mathbf{v}_I^\ell \in \mathbb{R}^I$ for $\ell \in L_1 \cup L_2$ by

$$\mathbf{v}_I^\ell = \begin{cases} \mathbf{x}_I^j - \mathbf{w}_I^j & \text{for } \ell = j \in L_1 \\ \mathbf{x}_I^j - \mathbf{x}'_I & \text{for } \ell = (j, \mathbf{x}'_I) \in L_2 \end{cases}$$

and scalars α_ℓ for $\ell \in L_1 \cup L_2$ by

$$\alpha_\ell = \begin{cases} V_{\mathrm{H}}^j \left(\mathbf{x}_I^j; u^j\right) + w_0^j + t^j & \text{for } \ell = j \in L_1 \\ V_{\mathrm{H}}^j \left(\mathbf{x}_I^j; u^j\right) - V_{\mathrm{H}}^j \left(\mathbf{x}'_I; u^j\right) & \text{for } \ell = (j, \mathbf{x}'_I) \in L_2. \end{cases}$$

By Claim A.6, there does not exist a price vector $\mathbf{p}_I$ such that $\mathbf{v}_I^\ell \cdot \mathbf{p}_I \leq \alpha_\ell$ for all $\ell \in L_1 \cup L_2$ with equality for all $\ell \in L_1$. The "if" direction of Fact A.7 therefore guarantees that there exist scalars λ_ℓ for $\ell \in L_1 \cup L_2$ with $\lambda_\ell \geq 0$ for all $\ell \in L_2$ such that

$$\sum_{\ell \in L_1 \cup L_2} \lambda_\ell \mathbf{v}_I^\ell = \mathbf{0} \quad \text{and} \quad \sum_{\ell \in L_1 \cup L_2} \lambda_\ell \alpha_\ell < 0.$$

By the definition of the scalars α_ℓ, we have that

$$\sum_{j \in J} \lambda_j \left(V_{\mathrm{H}}^j \left(\mathbf{x}_I^j; u^j\right) + w_0^j + t^j\right) + \sum_{j \in J} \sum_{\mathbf{x}'_I \in X_I^j} \lambda_{j, \mathbf{x}'_I} \left(V_{\mathrm{H}}^j \left(\mathbf{x}_I^j; u^j\right) - V_{\mathrm{H}}^j \left(\mathbf{x}'_I; u^j\right)\right) < 0.$$

Due the continuity of $V_{\mathrm{H}}^j \left(\hat{\mathbf{x}}_I^j; \cdot\right)$ for each agent j (Lemma A.2) and because $\mathbf{u}_{(m)} \to \mathbf{u}$ and $\mathbf{t}_{(m)} \to \mathbf{t}$ as $m \to \infty$, there must exist m such that

$$\sum_{j \in J} \lambda_j \left(V_{\mathrm{H}}^j \left(\mathbf{x}_I^j; u_{(m)}^j\right) + w_0^j + t_{(m)}^j\right) + \sum_{j \in J} \sum_{\mathbf{x}'_I \in X_I^j} \lambda_{j, \mathbf{x}'_I} \left(V_{\mathrm{H}}^j \left(\mathbf{x}_I^j; u_{(m)}^j\right) - V_{\mathrm{H}}^j \left(\mathbf{x}'_I; u_{(m)}^j\right)\right) < 0.$$

Defining scalars α'_ℓ for $\ell \in L_1 \cup L_2$ by

$$\alpha'_\ell = \begin{cases} V_{\mathrm{H}}^j \left(\mathbf{x}_I^j; u_{(m)}^j\right) + w_0^j + t_{(m)}^j & \text{for } \ell = j \in L_1 \\ V_{\mathrm{H}}^j \left(\mathbf{x}_I^j; u_{(m)}^j\right) - V_{\mathrm{H}}^j \left(\mathbf{x}'_I; u_{(m)}^j\right) & \text{for } \ell = (j, \mathbf{x}'_I) \in L_2, \end{cases}$$

we have that

$$\sum_{\ell \in L_1 \cup L_2} \lambda_\ell \mathbf{v}_I^\ell = \mathbf{0} \quad \text{and that} \quad \sum_{\ell \in L_1 \cup L_2} \lambda_\ell \alpha'_\ell < 0.$$

The "only if" implication of Fact A.7 therefore guarantees that there does not exist a price vector $\mathbf{p}_I$ such that $\mathbf{v}_I^\ell \cdot \mathbf{p}_I \leq \alpha'_\ell$ for all $\ell \in L_1 \cup L_2$ with equality for all $\ell \in L_1$. By Claim A.6, it follows that $\mathbf{t}_{(m)} \notin T(\mathbf{u}_{(m)})$—a contradiction. Hence, we can conclude that $\mathbf{t} \in T(\mathbf{u})$—as desired. ☐

To complete the proof of Claim A.4, we apply a topological fixed point argument.

Proof of Claim A.4. Consider the compact, convex set

$$Z = [\underline{M}, \overline{M}]^J \times \bigtimes_{j \in J} [u^j_{\min}, u^j_{\max}].$$

As $T(\mathbf{u}) \subseteq [\underline{M}, \overline{M}]^J$ for all $\mathbf{u} \in \bigtimes_{j \in J}[u^j_{\min}, u^j_{\max}]$, we can define a correspondence $\Phi : Z \rightrightarrows Z$ by

$$\Phi(\mathbf{t}, \mathbf{u}) = T(\mathbf{u}) \times \operatorname*{arg\,min}_{\hat{\mathbf{u}} \in \bigtimes_{j \in J}[u^j_{\min}, u^j_{\max}]} \left\{ \sum_{j \in J} t^j \hat{u}^j \right\}.$$

Claim A.5 guarantees that $T : \bigtimes_{j \in J}[u^j_{\min}, u^j_{\max}] \rightrightarrows \mathbb{R}^J$ is upper hemicontinuous and has compact, convex values. Statement (III) in Theorem A.1 ensures that the correspondence T has non-empty values. Because $\bigtimes_{j \in J}[u^j_{\min}, u^j_{\max}]$ is compact and convex, it follows that the correspondence Φ is upper hemicontinuous and has non-empty, compact, convex values as well. Hence, Kakutani's Fixed Point Theorem guarantees that Φ has a fixed point $(\mathbf{t}, \mathbf{u})$.

By construction, we have that $\mathbf{t} \in T(\mathbf{u})$ and that

$$u^j \in \operatorname*{arg\,min}_{\hat{u}^j \in [u^j_{\min}, u^j_{\max}]} t^j \hat{u}^j \tag{A.2}$$

for all agents j. It suffices to prove that $\mathbf{t} = \mathbf{0}$.

Let $(\mathbf{p}_I, (\mathbf{x}^j)_{j \in J})$ be a competitive equilibrium in the Hicksian economy for the profile $(u^j)_{j \in J}$ of utility levels with

$$\mathbf{p}_I \cdot (\mathbf{x}^j_I - \mathbf{w}^j_I) - V^j_{\mathrm{H}}\left(\mathbf{x}^j_I; u^j\right) - w^j_0 = t^j \tag{A.3}$$

for all agents j. As $u^j \geq u^j_{\min}$ and $V^j_{\mathrm{H}}\left(\mathbf{x}^j_I; \cdot\right)$ is decreasing for each agent j (by Lemma A.2), it follows from Equation (A.3) and the definition of K^j that

$$\begin{aligned} t^j &= \mathbf{p}_I \cdot (\mathbf{x}^j_I - \mathbf{w}^j_I) - V^j_{\mathrm{H}}\left(\mathbf{x}^j_I; u^j\right) - w^j_0 \\ &\geq \mathbf{p}_I \cdot (\mathbf{x}^j_I - \mathbf{w}^j_I) - V^j_{\mathrm{H}}\left(\mathbf{x}^j_I; u^j_{\min}\right) - w^j_0 \\ &\geq \mathbf{p}_I \cdot (\mathbf{x}^j_I - \mathbf{w}^j_I) - K^j \end{aligned} \tag{A.4}$$

for all agents j.

Next, we claim that $t^j \leq 0$ for all agents j. If $t^j > 0$, then Equation (A.2) would imply that $u^j = u^j_{\min}$. But as $\mathbf{t} \in T(\mathbf{u})$, it would follow that

$$\begin{aligned} t^j = \mathbf{p}_I \cdot (\mathbf{x}^j_I - \mathbf{w}^j_I) - V^j_{\mathrm{H}}\left(\mathbf{x}^j_I; u^j\right) - w^j_0 &\leq \mathbf{p}_I \cdot (\mathbf{w}^j_I - \mathbf{w}^j_I) - V^j_{\mathrm{H}}\left(\mathbf{w}^j_I; u^j\right) - w^j_0 \\ &= -V^j_{\mathrm{H}}\left(\mathbf{w}^j_I; u^j\right) - w^j_0 \\ &\leq \mathbf{p}_I \cdot (\mathbf{w}^j_I - \mathbf{w}^j_I) - V^j_{\mathrm{H}}\left(\mathbf{w}^j_I; u^j_{\min}\right) - w^j_0 \\ &= -V^j_{\mathrm{H}}\left(\mathbf{w}^j_I; u^j_{\min}\right) - w^j_0 \\ &= 0, \end{aligned}$$

where the first inequality holds since $(\mathbf{p}_I, (\mathbf{x}^j)_{j \in J})$ be a competitive equilibrium in the Hicksian economy for the profile $(u^j)_{j \in J}$ of utility levels, the second inequality holds holds since $V^j_{\mathrm{H}}\left(\mathbf{w}^j_I; \cdot\right)$ is decreasing (by Lemma A.2), and the last equality holds due to the definitions of V^j_{H} and $u^j_{\min}$. Thus, we can conclude that $t^j \leq 0$ must hold for all agents j.

As $(\mathbf{x}^j_I)_{j \in J}$ is the allocation of goods in a competitive equilibrium and $(\mathbf{w}^j_I)_{j \in J})$ is a feasible endowment profile, we have that $\sum_{j \in J} \mathbf{x}^j_I = \mathbf{y}_I = \sum_{j \in J} \mathbf{w}^j_I$ and hence that

$$\sum_{j \in J} \mathbf{p}_I \cdot (\mathbf{x}^j_I - \mathbf{w}^j_I) = 0 \geq \sum_{j \in J} t^j,$$

where the inequality holds because $t^j \leq 0$ for all agents j. It follows that for all agents j, we have that

$$t^j - \mathbf{p}_I \cdot (\mathbf{x}^j_I - \mathbf{w}^j_I) \leq \sum_{k \in J \smallsetminus \{j\}} (\mathbf{p}_I \cdot (\mathbf{x}^k_I - \mathbf{w}^k_I) - t^k) \leq \sum_{k \in J \smallsetminus \{j\}} K^k \leq \sum_{k \in J} K^k < K,$$

where the second inequality follows from Equation (A.4), the third inequality holds because $K^j \geq 0$, and the fourth inequality holds due to the definition of K. Hence, by Equation (A.3), we have that

$$-V^j_{\mathrm{H}}\left(\mathbf{x}^j_I; u^j\right) = w^j_0 + t^j - \mathbf{p}_I \cdot (\mathbf{x}^j_I - \mathbf{w}^j_I) < w^j_0 + K$$

for all agents j. Since utility is strictly increasing in the consumption of money, it follows that

$$u^j = U^j\left(-V^j_{\mathrm{H}}\left(\mathbf{x}^j_I; u^j\right), \mathbf{x}^j_I\right) < U^j\left(w^j_0 + K, \mathbf{x}^j_I\right) \leq u^j_{\max},$$

where the equality holds due to the definition of V^j_{H} and the second inequality holds due to the definition of $u^j_{\max}$. Equation (A.2) then implies that $t^j \geq 0$ for all agents j, so we must have that $t^j = 0$ for all agents j. □

Appendix B. Proof of Proposition 1

We actually prove a stronger statement.

Claim B.1. *Suppose that agent j demands at most one unit of each good and let $\mathbf{w}_I \in \{0,1\}^I$. A utility function U^j is a net substitutes utility function if for all money endowments w_0, price vectors $\mathbf{p}_I$, and $0 < \mu < \lambda$, whenever*

(i) $D^j_{\mathrm{M}}(\mathbf{p}_I, \mathbf{w}) = \{\mathbf{x}_I\}$,
(ii) $D^j_{\mathrm{M}}(\mathbf{p}_I + \lambda \mathbf{e}^i, \mathbf{w}) = \{\mathbf{x}'_I\}$,
(iii) $\{\mathbf{x}_I, \mathbf{x}'_I\} \subseteq D^j_{\mathrm{M}}(\mathbf{p}_I + \mu \mathbf{e}^i, \mathbf{w})$, *and*
(iv) $x'_i < x_i$,

we have that $x'_k \geq x_k$ for all goods $k \neq i$.

To complete the proof of the proposition from Claim B.1, we work in the setting of Claim B.1. Note that U^j is a gross substitutes utility function at endowment $\mathbf{w} = (w_0, \mathbf{w}_I)$ for all money endowments w_0 when $x'_k \geq x_k$ holds for all goods $k \neq i$ under Conditions (i) and (ii). This property clearly implies that $x'_k \geq x_k$ holds for all goods $k \neq i$ under Conditions (i), (ii), (iii), and (iv), and hence that U^j is net substitutes utility function by Claim B.1. The proposition therefore follows from Claim B.1.

It remains to prove Claim B.1. In the argument, we use the following characterization of substitutes valuations.

Fact B.2 (Theorems 2.1 and 2.4 in Fujishige and Yang, 2003; Theorems 3.9 and 4.10(iii) in Shioura and Tamura, 2015)**.** *Suppose that agent j demands at most one unit of each good. A valuation V^j is a substitutes valuation if and only if for all price vectors $\mathbf{p}_I$ with $|D^j(\mathbf{p}_I)| = 2$, writing $D^j(\mathbf{p}_I) = \{\mathbf{x}_I, \mathbf{x}'_I\}$, the difference $\mathbf{x}'_I - \mathbf{x}_I$ is a vector with at most one positive component and at most one negative component.*

Proof of Claim B.1. We prove the contrapositive. Suppose that U^j is not a net substitutes utility function. We show that there exists a money endowment w_0, a price vector $\mathbf{p}_I$, price increments $0 < \mu < \lambda$, and goods $i \neq k$ such that Conditions (i), (ii), (iii), and (iv) from the statement hold but $x'_k < x_k$.

By Remark 1, there exists a utility level u such that $V^j_{\mathrm{H}}(\cdot; u)$ is not a substitutes valuation. Hence, by Lemma 1 and the "if" direction of Fact B.2 for $V^j = V^j_{\mathrm{H}}(\cdot; u)$, there exists a price vector $\hat{\mathbf{p}}_I$ such that $|D^j_{\mathrm{H}}(\hat{\mathbf{p}}_I; u)| = 2$, and writing $D^j_{\mathrm{H}}(\hat{\mathbf{p}}_I; u) = \{\mathbf{x}_I, \mathbf{x}'_I\}$, the difference $\mathbf{x}'_I - \mathbf{x}_I$ has at least two positive components or at least two negative components. Without loss of generality, we can assume that the difference $\mathbf{x}'_I - \mathbf{x}_I$ has at least two negative components. Suppose that $x'_i < x_i$ (so Condition (iv) holds) and that $x'_k < x_k$, where $i, k \in I$ are distinct goods.

Define a money endowment w_0 by

$$w_0 = \hat{\mathbf{p}}_I \cdot (\mathbf{x}_I - \mathbf{w}_I) - V^j_{\mathrm{H}}(\mathbf{x}_I; u) = \hat{\mathbf{p}}_I \cdot (\mathbf{x}'_I - \mathbf{w}_I) - V^j_{\mathrm{H}}(\mathbf{x}'_I; u);$$

Fact 1 implies that $D^j_{\mathrm{M}}(\hat{\mathbf{p}}_I, \mathbf{w}) = \{\mathbf{x}_I, \mathbf{x}'_I\}$. Let $\mu > 0$ be such that

$$D^j_{\mathrm{M}}\left(\hat{\mathbf{p}}_I - \mu \mathbf{e}^i, \mathbf{w}\right), D^j_{\mathrm{M}}\left(\hat{\mathbf{p}}_I + \mu \mathbf{e}^i, \mathbf{w}\right) \subseteq \{\mathbf{x}_I, \mathbf{x}'_I\};$$

such a μ exists due to the upper hemicontinuity of D^j_{M}. Let $\mathbf{p}_I = \hat{\mathbf{p}}_I - \mu \mathbf{e}^i$, let $\lambda = 2\mu$, and let $\mathbf{p}'_I = \mathbf{p}_I + \lambda \mathbf{e}^i = \hat{\mathbf{p}}_I + \mu \mathbf{e}^i$.

By construction, we have that $\{\mathbf{x}_I, \mathbf{x}'_I\} \subseteq D^j_{\mathrm{M}}(\mathbf{p}_I + \mu \mathbf{e}^i, \mathbf{w}) = D^j_{\mathrm{M}}(\hat{\mathbf{p}}_I, \mathbf{w})$, so Condition (iii) holds. It remains to show that $D^j_{\mathrm{M}}(\mathbf{p}_I, \mathbf{w}) = \{\mathbf{x}_I\}$ and that $D^j_{\mathrm{M}}(\mathbf{p}'_I, \mathbf{w}) = \{\mathbf{x}'_I\}$. As j demands at most one unit of each good, we must have that $x_i = 1$ and that $x'_i = 0$. We divide into cases based on the value of w_i to show that

$$\begin{aligned} U^j\left(w_0 - \mathbf{p}_I \cdot (\mathbf{x}_I - \mathbf{w}_I), \mathbf{x}_I\right) &> U^j\left(w_0 - \mathbf{p}_I \cdot (\mathbf{x}'_I - \mathbf{w}_I), \mathbf{x}'_I\right) \\ U^j\left(w_0 - \mathbf{p}'_I \cdot (\mathbf{x}'_I - \mathbf{w}_I), \mathbf{x}'_I\right) &> U^j\left(w_0 - \mathbf{p}'_I \cdot (\mathbf{x}_I - \mathbf{w}_I), \mathbf{x}_I\right). \end{aligned} \tag{B.1}$$

Case 1: $w_i = 0$. In this case, we have that

$$\begin{aligned} U^j\left(w_0 - \mathbf{p}_I \cdot (\mathbf{x}_I - \mathbf{w}_I), \mathbf{x}_I\right) &> U^j\left(w_0 - \hat{\mathbf{p}}_I \cdot (\mathbf{x}_I - \mathbf{w}_I), \mathbf{x}_I\right) \\ &= U^j\left(w_0 - \hat{\mathbf{p}}_I \cdot (\mathbf{x}'_I - \mathbf{w}_I), \mathbf{x}'_I\right) \\ &= U^j\left(w_0 - \mathbf{p}_I \cdot (\mathbf{x}'_I - \mathbf{w}_I), \mathbf{x}'_I\right), \end{aligned}$$

where the inequality holds because $p_i < \hat{p}_i$ and $x_i > w_i$, the first equality holds because $\{\mathbf{x}_I, \mathbf{x}'_I\} \subseteq D^j_{\mathrm{M}}\left(\hat{\mathbf{p}}_I, \mathbf{w}_I\right)$, and the second equality holds because $x'_i = w_i$. Similarly, we have that

$$\begin{aligned} U^j\left(w_0 - \mathbf{p}'_I \cdot (\mathbf{x}_I - \mathbf{w}_I), \mathbf{x}_I\right) &< U^j\left(w_0 - \hat{\mathbf{p}}_I \cdot (\mathbf{x}_I - \mathbf{w}_I), \mathbf{x}_I\right) \\ &= U^j\left(w_0 - \hat{\mathbf{p}}_I \cdot (\mathbf{x}'_I - \mathbf{w}_I), \mathbf{x}'_I\right) \\ &= U^j\left(w_0 - \mathbf{p}'_I \cdot (\mathbf{x}'_I - \mathbf{w}_I), \mathbf{x}'_I\right), \end{aligned}$$

where the inequality holds because $p'_i > \hat{p}_i$ and $x_i > w_i$, the first equality holds because $\{\mathbf{x}_I, \mathbf{x}'_I\} \subseteq D^j_{\mathrm{M}}\left(\hat{\mathbf{p}}_I, \mathbf{w}_I\right)$, and the second equality holds because $x'_i = w_i$.

Case 2: $w_i = 1$. In this case, we have that

$$\begin{aligned} U^j\left(w_0 - \mathbf{p}_I \cdot (\mathbf{x}'_I - \mathbf{w}_I), \mathbf{x}'_I\right) &< U^j\left(w_0 - \hat{\mathbf{p}}_I \cdot (\mathbf{x}'_I - \mathbf{w}_I), \mathbf{x}'_I\right) \\ &= U^j\left(w_0 - \hat{\mathbf{p}}_I \cdot (\mathbf{x}_I - \mathbf{w}_I), \mathbf{x}_I\right) \\ &= U^j\left(w_0 - \mathbf{p}_I \cdot (\mathbf{x}_I - \mathbf{w}_I), \mathbf{x}_I\right) \end{aligned}$$

where the inequality holds because $p_i < \hat{p}_i$ and $x'_i < w_i$, the first equality holds because $\{\mathbf{x}_I, \mathbf{x}'_I\} \subseteq D^j_{\mathrm{M}}\left(\hat{\mathbf{p}}_I, \mathbf{w}_I\right)$, and the second equality holds because $x_i = w_i$. Similarly, we have that

$$\begin{aligned} U^j\left(w_0 - \mathbf{p}'_I \cdot (\mathbf{x}'_I - \mathbf{w}_I), \mathbf{x}'_I\right) &> U^j\left(w_0 - \hat{\mathbf{p}}_I \cdot (\mathbf{x}'_I - \mathbf{w}_I), \mathbf{x}'_I\right) \\ &= U^j\left(w_0 - \hat{\mathbf{p}}_I \cdot (\mathbf{x}_I - \mathbf{w}_I), \mathbf{x}_I\right) \\ &= U^j\left(w_0 - \mathbf{p}'_I \cdot (\mathbf{x}_I - \mathbf{w}_I), \mathbf{x}_I\right), \end{aligned}$$

where the inequality holds because $p'_i > \hat{p}_i$ and $x'_i < w_i$, the first equality holds because $\{\mathbf{x}_I, \mathbf{x}'_I\} \subseteq D^j_{\mathrm{M}}\left(\hat{\mathbf{p}}_I, \mathbf{w}_I\right)$, and the second equality holds because $x_i = w_i$.

As $\mathbf{w}_I \in \{0,1\}^I$, the cases exhaust all possibilities. Hence, we have proven that Equation (B.1) must hold. As $D^j_{\mathrm{M}}\left(\mathbf{p}_I, \mathbf{w}\right), D^j_{\mathrm{M}}\left(\mathbf{p}'_I, \mathbf{w}\right) \subseteq \{\mathbf{x}_I, \mathbf{x}'_I\}$, we must have that $D^j_{\mathrm{M}}\left(\mathbf{p}_I, \mathbf{w}\right) = \{\mathbf{x}_I\}$ and that $D^j_{\mathrm{M}}\left(\mathbf{p}'_I, \mathbf{w}\right) = \{\mathbf{x}'_I\}$—so Conditions (i) and (ii) hold, as desired. $\square$

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Appendix C. Proof of Fact 1

We begin by proving two technical claims.

Claim C.1. *Let* $\mathbf{w}$ *be an endowment and let* u *be a utility level. If*

$$D^j_{\mathrm{M}}(\mathbf{p}_I, \mathbf{w}) \neq \varnothing \qquad \text{and} \qquad u = \max_{\mathbf{x} \in X^j | \mathbf{p} \cdot \mathbf{x} \leq \mathbf{p} \cdot \mathbf{w}} U^j(\mathbf{x}), \tag{C.1}$$

then we have that

$$\mathbf{p} \cdot \mathbf{w} = \min_{\mathbf{x} \in X^j | U^j(\mathbf{x}) \geq u} \mathbf{p} \cdot \mathbf{x}$$

and that $D^j_{\mathrm{M}}(\mathbf{p}_I, \mathbf{w}) \subseteq D^j_{\mathrm{H}}(\mathbf{p}_I; u)$.

Proof. Letting $\mathbf{x}'_I \in D^j_{\mathrm{M}}(\mathbf{p}_I, \mathbf{w})$ be arbitrary and $x'_0 = w_0 - \mathbf{p}_I \cdot (\mathbf{x}'_I - \mathbf{w}_I)$, we have that $U^j(\mathbf{x}') = u$ and that $\mathbf{p} \cdot \mathbf{x}' \leq \mathbf{p} \cdot \mathbf{w}$ by construction. It follows that

$$\mathbf{p} \cdot \mathbf{w} \geq \min_{\mathbf{x} \in X^j | U^j(\mathbf{x}) \geq u} \mathbf{p} \cdot \mathbf{x}.$$

Suppose for the sake of deriving a contradiction that there exists $\mathbf{x}'' \in X^j$ with $\mathbf{p} \cdot \mathbf{x}'' < \mathbf{p} \cdot \mathbf{w}$ and $U^j(\mathbf{x}'') \geq u$. Then, we have that $x''_0 < w_0 + \mathbf{p}_I \cdot (\mathbf{x}''_I - \mathbf{w}_I)$; write $x'''_0 = w_0 + \mathbf{p}_I \cdot (\mathbf{x}''_I - \mathbf{w}_I)$, so $x'''_0 > x''_0$. Since U^j is strictly increasing in consumption of money, it follows that $U^j(x'''_0, \mathbf{x}''_I) > u$—contradicting Equation (C.1) as $x'''_0 + \mathbf{p}_I \cdot \mathbf{x}''_I = \mathbf{p} \cdot \mathbf{w}$. Hence, we can conclude that

$$\mathbf{p} \cdot \mathbf{w} = \min_{\mathbf{x} \in X^j | U^j(\mathbf{x}) \geq u} \mathbf{p} \cdot \mathbf{x}.$$

Since $U^j(\mathbf{x}') = u$ and $\mathbf{p} \cdot \mathbf{x}' = \mathbf{p} \cdot \mathbf{w}$, it follows that $\mathbf{x}'_I \in D^j_{\mathrm{H}}(\mathbf{p}_I; u)$. Since $\mathbf{x}'_I \in D^j_{\mathrm{M}}(\mathbf{p}_I, \mathbf{w})$ was arbitrary, we can conclude that $D^j_{\mathrm{M}}(\mathbf{p}_I, \mathbf{w}) \subseteq D^j_{\mathrm{H}}(\mathbf{p}_I; u)$. □

Claim C.2. *Let* $\mathbf{w}$ *be an endowment and let* u *be a utility level. If*

$$\mathbf{p} \cdot \mathbf{w} = \min_{\mathbf{x} \in X^j | U^j(\mathbf{x}) \geq u} \mathbf{p} \cdot \mathbf{x}, \tag{C.2}$$

then we have that

$$u = \max_{\mathbf{x} \in X^j | \mathbf{p} \cdot \mathbf{x} \leq \mathbf{p} \cdot \mathbf{w}} U^j(\mathbf{x})$$

and that $D^j_{\mathrm{H}}(\mathbf{p}_I; u) \subseteq D^j_{\mathrm{M}}(\mathbf{p}_I, \mathbf{w})$.

Proof. Let $\mathbf{x}'_I \in D^j_{\mathrm{H}}(\mathbf{p}_I; u)$ be arbitrary and $x'_0 = -V^j_{\mathrm{H}}(\mathbf{x}'_I; u)$. We have that $U^j(\mathbf{x}') \geq u$ and that $\mathbf{p} \cdot \mathbf{x}' = \mathbf{p} \cdot \mathbf{w}$ by construction. It follows that

$$u \leq \max_{\mathbf{x} \in X^j | \mathbf{p} \cdot \mathbf{x} \leq \mathbf{p} \cdot \mathbf{w}} U^j(\mathbf{x}).$$

We next show that

$$u = \max_{\mathbf{x} \in X^j | \mathbf{p} \cdot \mathbf{x} \leq \mathbf{p} \cdot \mathbf{w}} U^j(\mathbf{x}).$$

Suppose for sake of deriving a contradiction that there exists $\mathbf{x}'' \in X^j$ with $\mathbf{p} \cdot \mathbf{x}'' \leq \mathbf{p} \cdot \mathbf{w}$ and $U^j(\mathbf{x}'') > u$. By the definition of V^j_{H}, we have that $x''_0 = -V^j_{\mathrm{H}}(\mathbf{x}''_I; U^j(\mathbf{x}''_I))$, and since $V^j_{\mathrm{H}}(\mathbf{x}''_I; \cdot)$ is strictly decreasing (by Lemma A.2), it follows that $x''_0 > -V^j_{\mathrm{H}}(\mathbf{x}''_I; u)$. Letting $x'''_0 = -V^j_{\mathrm{H}}(\mathbf{x}''_I; u)$, we have that $x'''_0 + \mathbf{p}_I \cdot \mathbf{x}''_I < \mathbf{p} \cdot \mathbf{w}$, which contradicts Equation (C.2) as $U^j(x'''_0, \mathbf{x}''_I) = u$. Hence, we can conclude that

$$u = \max_{\mathbf{x} \in X^j \mid \mathbf{p} \cdot \mathbf{x} \leq \mathbf{p} \cdot \mathbf{w}} U^j(\mathbf{x}).$$

Since $U^j(\mathbf{x}'_I) = u$ and $\mathbf{p} \cdot \mathbf{x}'_I = \mathbf{p} \cdot \mathbf{w}$, it follows that $\mathbf{x}'_I \in D^j_{\mathrm{M}}(\mathbf{p}_I, \mathbf{w})$. Since $\mathbf{x}'_I \in D^j_{\mathrm{H}}(\mathbf{p}_I; u)$ was arbitrary, we can conclude that $D^j_{\mathrm{H}}(\mathbf{p}_I; u) \subseteq D^j_{\mathrm{M}}(\mathbf{p}_I, \mathbf{w})$. □

Let $\mathbf{w}$ be an endowment and let u be a utility level. By Claims C.1 and C.2, Conditions (C.1) and (C.2) are equivalent, and under these equivalent conditions, we have that $D^j_{\mathrm{M}}(\mathbf{p}_I, \mathbf{w}) \subseteq D^j_{\mathrm{H}}(\mathbf{p}_I; u)$ and that $D^j_{\mathrm{H}}(\mathbf{p}_I; u) \subseteq D^j_{\mathrm{M}}(\mathbf{p}_I, \mathbf{w})$. Hence, we must have that $D^j_{\mathrm{M}}(\mathbf{p}_I, \mathbf{w}) = D^j_{\mathrm{H}}(\mathbf{p}_I; u)$ under the equivalent Conditions (C.1) and (C.2)—as desired.

Appendix D. Proof of Fact 3

D.1. **Preliminaries.** We use the concept of a pseudo-equilibrium price vector.

Definition D.1 (Milgrom and Strulovici, 2009)**.** Suppose that utility is transferable. A *pseudo-equilibrium price vector* is a price vector $\mathbf{p}_I$ such that

$$\mathbf{y}_I \in \mathrm{Conv}\left(\sum_{j \in J} D^j(\mathbf{p}_I)\right).$$

There is a connection between pseudo-equilibrium price vectors, competitive equilibria, and the existence problem.

Fact D.2 (Theorem 18 in Milgrom and Strulovici, 2009; Lemma 2.19 in Baldwin and Klemperer, 2019)**.** *If utility is transferable and the total endowment of goods is such that a competitive equilibrium exists, then, for each pseudo-equilibrium price vector $\mathbf{p}_I$, there exists an allocation $(\mathbf{x}^j_I)_{j \in J}$ such that $\mathbf{p}_I$ and $(\mathbf{x}^j_I)_{j \in J}$ comprise a competitive equilibrium.*

The nonexistence of competitive equilibria may therefore be demonstrated by using the contrapositive of Fact D.2.

D.2. **Proof of Fact 3.** By Fact B.2, there exists a price vector $\mathbf{p}_I$ such that $D^j(\mathbf{p}_I) = \{\mathbf{x}'_I, \mathbf{x}'_I + \mathbf{g}\}$, where $\mathbf{g}$ has at least two positive components or at least two negative components. Identify I with $\{1, \ldots, |I|\}$ and without loss of generality assume that $g_1, g_2 < 0$. Because agent j demands at most one unit of each good, we know that $\mathbf{x}'_I, \mathbf{x}'_I + \mathbf{g} \in \{0, 1\}^{|I|}$, so $\mathbf{g} \in \{-1, 0, 1\}^{|I|}$.

Let $k \in J \smallsetminus \{j\}$ be arbitrary. For $j' \in J \smallsetminus \{j,k\}$, define $V^{j'} : \{0,1\}^I \to \mathbb{R}$ by

$$V^{j'}(\mathbf{x}_I) = \mathbf{p}_I \cdot \mathbf{x}_I - \sum_{i=1}^{|I|} x_i,$$

which is a substitutes valuation as it is additive (see, e.g., Example 2 in Danilov, Koshevoy, and Lang (2003)). By construction, we have that $D^{j'}(\mathbf{p}_I) = \{\mathbf{0}\}$ for all $j' \in J \smallsetminus \{j,k\}$—which is the only property of the valuations that we use in the sequel.

To define V^k, consider the valuation $V : \{0,1\}^{|I|} \to \mathbb{R}$ defined by

$$V(\mathbf{x}_I) = \begin{cases} 0 & \text{if } x_1 + x_2 = 0 \\ 2 & \text{if } x_1 + x_2 > 0 \end{cases},$$

which is a substitutes valuation as it is a unit-demand valuation in the sense of Gul and Stacchetti (1999). Now define $V^k : \{0,1\}^{|I|} \to \mathbb{R}$ by

$$V^k(\mathbf{x}_I) = V(\mathbf{x}_I) + \mathbf{p}_I \cdot \mathbf{x}_I - x_1 - x_2.$$

As V^k is the sum of a substitutes valuation and an additive valuation, it is also a substitutes valuation (by, e.g., Remark 1 on page 289 of Danilov, Koshevoy, and Lang (2003)). By construction, we have that

$$D^k(\mathbf{p}_I) = \operatorname*{arg\,max}_{\mathbf{x}_I \in \{0,1\}^I} \{V(\mathbf{x}_I) - x_1 - x_2\} = \left\{\mathbf{x}_I \in \{0,1\}^{|I|} \,\middle|\, x_1 + x_2 = 1\right\}.$$

Observing that $\mathbf{e}^2 \in D^k(\mathbf{p}_I)$ and considering the vectors from $\mathbf{e}^2$ to other elements of the demand set, we can write $D^k(\mathbf{p}_I)$ as

$$D^k(\mathbf{p}_I) = \mathbf{e}^2 + \left\{\alpha_2(\mathbf{e}^1 - \mathbf{e}^2) + \sum_{\ell=3}^{|I|} \alpha_\ell \mathbf{e}^\ell \,\middle|\, \alpha_\ell \in \{0,1\} \text{ for } 2 \le \ell \le |I|\right\}.$$

Combining this with $D^j(\mathbf{p}_I)$, and recalling $D^{j'}(\mathbf{p}_I) = \{\mathbf{0}\}$ for all $j' \in J \smallsetminus \{j,k\}$, we conclude that

$$\sum_{j' \in J} D^{j'}(\mathbf{p}_I) = \mathbf{x}'_I + \mathbf{e}^2 + \left\{\alpha_1 \mathbf{g} + \alpha_2(\mathbf{e}^1 - \mathbf{e}^2) + \sum_{\ell=3}^{|I|} \alpha_\ell \mathbf{e}^\ell \,\middle|\, \alpha_\ell \in \{0,1\} \text{ for } 1 \le \ell \le |I|\right\}.$$

The convex hull of this set can be expressed very similarly, but where the weights α_ℓ are allowed to lie in $[0,1]$.

Since $\mathbf{x}'_I, \mathbf{x}'_I + \mathbf{g} \in \{0,1\}^{|I|}$, we have that if $g_i = 1$ (resp. $g_i = -1$), then $x'_i = 0$ (resp. $x'_i = 1$). Taking

$$\alpha_\ell = \begin{cases} \dfrac{|g_\ell|}{2} & \text{if } g_\ell \neq 0 \\ 1 - x'_\ell & \text{if } g_\ell = 0 \end{cases}$$

for $1 \leq \ell \leq |I|$, we have that

$$x'_i + \alpha_1 g_i + \alpha_i = 1 - \frac{1}{2} + \frac{1}{2} = 1 \qquad \text{for all } 1 \leq \ell \leq |I| \text{ with } g_i = -1$$
$$x'_i + \alpha_1 g_i + \alpha_i = x'_i + 0 + (1 - x'_i) = 1 \qquad \text{for all } 1 \leq \ell \leq |I| \text{ with } g_i = 0$$
$$x'_i + \alpha_1 g_i + \alpha_i = 0 + \frac{1}{2} + \frac{1}{2} = 1 \qquad \text{for all } 1 \leq \ell \leq |I| \text{ with } g_i = 1.$$

Since $g_1, g_2 < 0$, we have that $g_1 = g_2 = -1$, and hence that $x'_1 = x'_2 = 1$. It follows that

$$\mathbf{x}'_I + \mathbf{e}^2 + \alpha_1 \mathbf{g} + \alpha_2(\mathbf{e}^1 - \mathbf{e}^2) + \sum_{\ell=3}^{|I|} \alpha_\ell \mathbf{e}^\ell = \mathbf{y}_I.$$

As $\alpha_\ell \in [0, 1]$ for all $1 \leq \ell \leq |I|$, we therefore have that $\mathbf{y}_I \in \operatorname{Conv}\left(\sum_{j' \in J} D^{j'}(\mathbf{p}_I)\right)$, so $\mathbf{p}_I$ is a pseudo-equilibrium price vector. But as $\alpha_1 \in (0, 1)$ and the vectors $\mathbf{g}, \mathbf{e}^1 - \mathbf{e}^2, \mathbf{e}^3, \ldots, \mathbf{e}^{|I|}$ are linearly independent, we have that $\mathbf{y}_I \notin \sum_{j' \in J} D^{j'}(\mathbf{p}_I)$, so there is no competitive equilibrium at $\mathbf{p}_I$. Therefore, by the contrapositive of Fact D.2, no competitive equilibrium can exist.